THE UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR GRANT V. ZACKOWSKI

TITLE OF THESIS MINIMUM SENSITIVITY DESIGN OF VTOL AIRCRAFT CONTROLLER USING POLE ASSIGNMENT

DEGREE FOR WHICH THESIS WAS PRESENTED MASTER OF SCIENCE

YEAR THIS DEGREE GRANTED 1977

THE UNIVERSITY OF ALBERTA

MINIMUM SENSITIVITY DESIGN OF VTOL AIRCRAFT CONTROLLER USING POLE ASSIGNMENT

by

(C) GRANT V. ZACKOWSKI

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF ELECTRICAL ENGINEERING

EDMONTON, ALBERTA

FALL, 1977

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled Minimum Sensitivity Design of VTOL Aircraft Controller Using Pole Assignment submitted by Grant V. Zackowski in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

A techinque is described for the design of a multivariable controller for a VTOL (Vertical Takeoff and Landing) helicopter, based on a recent technique incorporating pole assignment and minimum eigenvalue sensitivity. A fourth order state space model of the helicopter available in the literature and used by earlier researchers is used in the present design also. Since the elements of the system matrix of the model do change over varying flight conditions, the minimum sensitivity design approach used in this thesis has a decided advantage over many previous design procedures which neglected the parameter variations and designed fixed feedback controllers or resorted to more complex adaptive techniques.

ACKNOWLEDGEMENT

I wish to express my deep appreciation and thanks to my supervisor, Dr. V. Gourishankar and his assistant, Dr. P. Kudva, for their many hours of moral support, help and assistance in the preparation of this thesis.

A special thanks must also be extended to my wife for her patience, support and encouragement over the past six years, without which, the work presented here may not have been possible.

The financial support provided by the Department of Electrical Engineering and the Canadian Armed Forces is gratefully acknowledged.

TABLE OF CONTENTS

Page

CHAPTER

LIST OF TABLES

LIST OF FIGURES

Figure		Page

LIST OF SYMBOLS

Symbol	Description
$\|\ \|$	absolute value
$\sum$	summation
Δ	change in
δ	perturbation in
$\rightarrow$	approaches or takes the value of
α	characteristic equation coefficient
$<$	less than
$>$	greater than
λ	eigenvalue or eingenvector
ω	frequency
ζ	damping ratio
$\equiv$	identically equal to
$\hat{}$	used for matrix identification
$\sim$	used for matrix identification
∂	derivative
s	pole(eigenvalue) location
$\angle$	angle

CHAPTER (1)

INTRODUCTION

1.1 History of Vertical Take-off and Landing (VTOL) Aircraft

One of the first concepts of VTOL* aircraft was recorded by Leonardo da Vinci in 1483. It was not until near the end of the eighteenth century when the first physical model of a VTOL aircraft was made by Launoy and Bienvena [15].**

During the next one hundred and twenty years there were many concepts proposed and implemented on numerous VTOL aircraft configurations in many countries of the world. In 1907 Breguet of France designed and built a VTOL rotary wing aircraft which was successful in lifting itself and a pilot off the ground. Although this aircraft could fly, it could do so only when tethered to the ground since the pilot had absolutely no directional control over the aircraft. It was this lack of control which plagued every inventor and engineer who pioneered in the development of the VTOL aircraft.

1.2 Development of Control Systems

During the first World War, Vonkarman and Petroczy produced a capative helicopter used only for army observation duties. This helicopter could be stabilized only by use of its mooring cables. From 1910 to 1920 the advances seen in the development of the VTOL aircraft was almost nil.

---

* Hereafter in this thesis, VTOL will be understood to mean specifically helicopter.

** Numbers in rectangular brackets refer to referances listed under Bibliography.

The next ten to fifteen years saw many experimental designs of helicopters. During this period of time the designers used a "direct" method for pilot control over the various modes of operation of the aircraft. This is shown in figure 1 for only one control mode, forward flight. Although this was a feasible method of control, it was not practical over even a short period of time because constant and absolute pilot attention was required.

In 1935-1936 Breguent-Dorand of France was able to maintain a helicopter airborne for a reasonable time which exhibited good control characteristics. The Breguet-Dorand helicopter provided,

i) cyclic pitch control - to govern horizontal flight;

ii) collective pitch control - to govern vertical flight; and

iii) differential collective control - to govern yaw motion.

Rather than the direct control, these controls were connected by systems of bell-cranks, levers and plates from the pilot controls to the respective controlled mode of operation. This type of control was used and improved upon over the subsequent years.

It was not until the introduction of modern control theory that a significant change was seen in the design of controls for the VTOL aircraft.

1.3 Application of Modern Control Theory to VTOL Aircraft Controls

With the introduction of modern control theory, the control of the VTOL aircraft was greatly increased and improved upon. This was possible since now the pilot was able to divert his attention to tasks other than strictly that of flying the aircraft.

In the modern day helicopter the input commands of the pilot

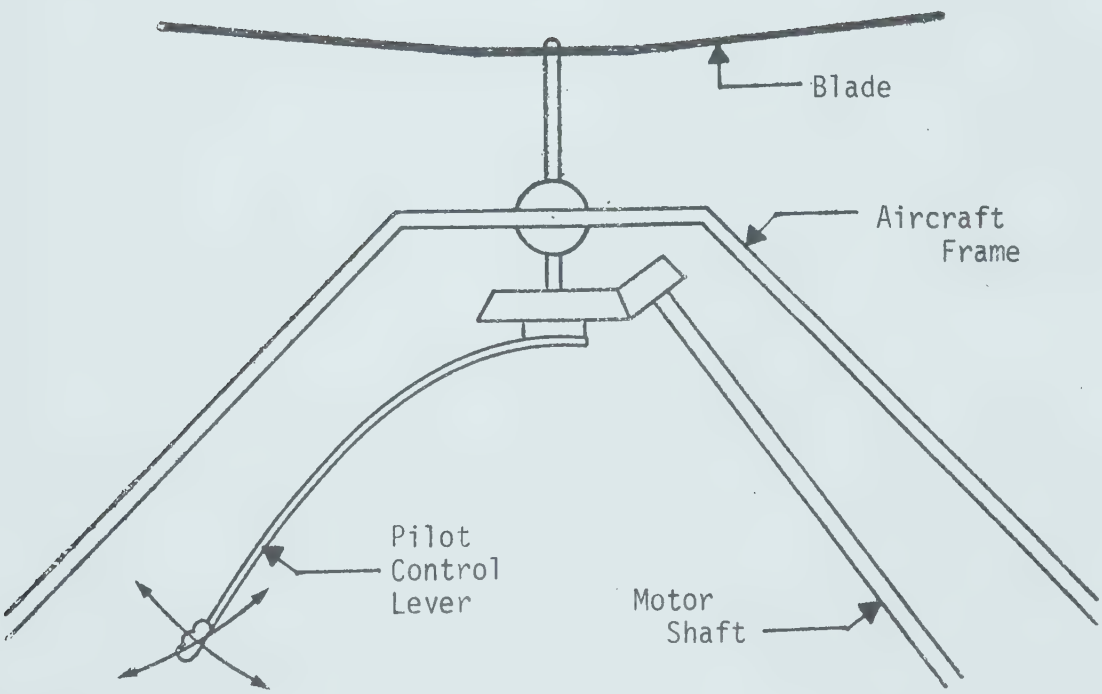

Figure 1. Direct Control

are supplemented by means of feedback from the output of the control system. This feedback can be accomplished electronically in a number of ways.

A constant gain feedback controller, which is nearly always preferred because of its simplicity and cost, is one means of providing the necessary control and stability augmentation. With the increasing advancements in technology more sophisticated control systems are being proposed. For instance adaptive controllers are used, which in many cases, require on-board computers for continual updating of the feedback matrix to ensure stability of the system [1]. These systems while being very promising, are also very expensive to implement on existing VTOL aircraft. Until the recently proposed systems can be implemented at a more resonable cost there may still be ways of re-designing the existing systems so that they are more acceptable to the aircraft industry both from the point of view of performance and cost.

What follows in subsequent chapters outlines a new procedure for designing a simpler and less costly constant gain feedback controller. The method results in a more stable system with better response characteristics over the flight regime than those of some previous design procedures which also employ constant gain feedback controllers.

1.4 Scope of Thesis

The aim of research reported in this thesis is to show the feasibility of using pole assignment and minimum sensitivity design technique to develop a controller for use in a VTOL aircraft.

In Chapter 2, a brief outline of the method to be used will be given. Also an arbitrary second-order linear time invarient multivariable

system will be used to demonstrate the approach used. The results will then be compared to those obtained using an aribitrary feedback controller.

Chapter 3 shows the results obtained using the minimum sensitivity approach when applied to a helicopter. In this study the model used by K.S. Narendra and S.S. Tripathi [1] is used. An arbitrary controller is then used on the same system and the results compared to those obtained using the minimum sensitivity design approach.

Chapter 4 discusses the introduction of integral control along with minimum sensitivity design in developing a controller for improving the transient and steady state performance of the VTOL aircraft.

In Chapter 5, the results are summarized and conclusions stated.

CHAPTER (2)

POLE ASSIGNMENT WITH MINIMUN EIGENVALUE SENSITIVITY TO PLANT PARAMETER VARIATIONS

2.1 Introduction

In this chapter a procedure is described for assigning the closed loop poles of a feedback control system to specified locations in the complex frequency plane in such a way that the poles have minimum sensitivity to plant parameter variations. The discussion follows closely the work of Gourishanker and Ramar [24]. This design procedure is used later in this thesis for designing feedback controllers for VTOL aircraft.

In the design of control systems, mathematical models of the plant or process to be controlled are used. In many cases the parameters of the plant are subject to variations caused by several factors such as environmental changes, external control inputs, ageing of components, etc. In many cases, even small changes in the values of the plant parameters may affect the system behaviour appreciably. Since the design of controllers is usually based on nominal values of plant parameters it becomes necessary to study the effect of parameter variations on system behaviour. The sensitivity of system behaviour to variations in plant parameters as reflected in the mathematical model has engaged the attention of reasearchers in recent years. One measure of this is known as eigenvalue sensitivity which is defined as the sensitivity of the closed-loop poles of a system to variations in plant parameters. Although eigenvalue sensitivity is a less direct measure of system performance it is being recognized more and more as a very useful measure of system performance [18, 24, 26]. One advantage of using eigenvalue sensitivity is that it can be easily

computed. It is well known that all the closed-loop poles can be assigned exactly to desired locations when all the state variables are available for feedback [28]. However, the solution is not unique in that more than one feedback controller can be designed to achieve the same objective as far as pole locations are concerned. Different feedback controllers may result in different behaviour of the system from some other point of view. This design freedom can be used to satisfy some other design criterion without altering the desired pole locations.

In the procedure described in this chapter this design freedom is used to minimize the sensitivity of the assigned closed-loop poles with respect to variations in the parameters of the plant. In other words, pole assignment and insensitive design are combined. The closed-loop poles are specified a priori in order to achieve a certain transient response and the feedback controller is designed to achieve these pole locations. The sensitivity of these poles to parameter variations will be a minimum.

2.2. Design of a Unity Rank Feedback Controller for Pole Assignment

Consider a linear time invarient multivariable system represented by

$$\overset{\circ}{x} = Ax + Bu \text{ -- } (2\text{-}1)$$

where x is an n-dimensional state vector, u is an r-dimensional input vector, A is the n x n system matrix and B is the nxr input matrix. This multi-input system is reduced to an equivalent single input system by defining a new scalar input u' as

$$u = qu' + R \text{ ------------------------------------- } (2\text{-}2)$$

where

$$q = \left[1, q_1, \ldots, q_r\right]^T \text{ -------------------------- (2-3)}$$

is an r-dimensional column vector, and R is the r-dimensional external input control vector. Then using (2-2) equation (2-1) becomes

$$\overset{\circ}{x} = Ax + Bqu + BR \text{ ------------------------------ (2-4)}$$

Now a feedback control

$$u' = kx \text{ -- (2-5)}$$

is designed such that the poles of the closed-loop system

$$\left.\begin{aligned} \overset{\circ}{x} &= (A + Bqk)x + BR \\ \overset{\circ}{x} &= (A + b_q k)x + BR \\ \overset{\circ}{x} &= \hat{A} + BR \end{aligned}\right\} \text{ -------------------------- (2-6)}$$

where

$$b_q = Bq$$
$$\hat{A} = A + b_q k$$

are at the specified locations $s_1, s_2, \ldots s_n$. Note that k is an n-dimensional row vector

$$k = \left[k_{11}\ k_{12}\ \ldots\ k_{1n}\right] \text{ -------------------------- (2-7)}$$

and is a function of the r-dimensional vector q. In general q is chosen arbitrarily. In this thesis q wil be chosen to minimize the sensitivity of the eigenvalues of the closed-loop system. Once q is chosen the vector k is determined for the specified pole locations. The rxn unity-rank feedback matrix K for the given closed-loop system (2-6) is easily obtained as

$$K = qk \text{ -- (2-8)}$$

2.3 Eigenvalue Sensitivity

Since we are using unity-rank feedback, the vector q will be chosen to ensure minimum sensitivity of the eigenvalues of the closed-loop system, thus resulting in the minimum sensitivity feedback controller as defined by (2-8).

Following Morgan [26] the sensitivity $S^i_{j\ell}$ of any pole (eigenvalue) s_i of the closed-loop system matrix $\hat{A}$ in equation (2-6) with respect to a "small" variation in the element $a_{j\ell}$ of the open-loop system matrix A is defined as

$$S^i_{j\ell} = \frac{\partial s_i}{\partial a_{j\ell}} = \frac{1}{g'(s_i)} \, tr\left[R(s_i) \, \frac{\partial \hat{A}}{\partial a_{j\ell}}\right] \quad \text{------------} \quad (2\text{-}9)$$

where

$g'(s_i)$ is the derivative of the closed-loop system characteristic polynominal with respect to s evaluated at $s = s_i$;

$R(s_i)$ is the adjoint$(s_i I - \hat{A})$, I is the nxn Identity matrix; and

tr denotes the trace of a matrix.

As can be seen from (2-6), the closed-loop system matrix $\hat{A}$, is a function of q and k; k is a function of q and therefore the sensitivity $S^i_{j\ell}$ is directly a function of q. In order to minimize the effect of the variations in the parameter $a_{j\ell}$ on $S^i_{j\ell}$ a sensitivity functional is formulated [26] as

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{n} (|S^i_{j\ell}|)^2 \quad \text{--------------------} \quad (2\text{-}10)$$

J obviously is a function of q. The objective is to determine q such that J is a minimum. Note that (2-10) is general in that it is applicable when all the elements of A are subject to variation. Now to facilitate the minimization of J with respect to q, the sensitivity functional, J should be evaluted in terms of q. This is accomplished if the sensitivity, $S^{i}_{j\ell}$, can be determined as only a function of q. One method of achieving this is to transform the matrix pair $(\hat{A},b_q)$ to phase-variable or canonical form. By using a method reported in the literature [19], a transformation matrix P(q) is determined which transforms the matrix pair $(\hat{A},b_q)$ to phase-variable form, i.e the transformation

$$\left.\begin{aligned} x &= P(q)^{-1}\, z \\ \text{or} \qquad z &= P(q)\, x \end{aligned}\right\} \text{------------------------------} \qquad (2\text{-}11)$$

is obtained.

Now using the transformation equation (2-11) one obtains for the transformed closed-looped system equation

$$\left.\begin{aligned} \overset{\circ}{z} &= P(q)(A + b_q k)P(q)^{-1}\, z \;+\; P(q)BR \\ \text{or} \qquad \overset{\circ}{z} &= A_o z \;+\; P(q)BR \end{aligned}\right\} \text{-------} \qquad (2\text{-}12)$$

where

$$A_o = P(q)(A + b_q k)P(q)^{-1} \quad \text{--------------------} \qquad (2\text{-}13)$$

The preselection of the desired poles of the closed-loop system matrix $\hat{A}$ determines the elements of A_o, where A_o in canonical form is written as

$$A_o = \begin{bmatrix} 0 & 1 & 0 \,\ldots\ldots\ldots & 0 \\ 0 & 0 & 1 \,\ldots\ldots\ldots & 0 \\ \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 \,..\, & 0 \,\ldots\, 0 & 1 \\ -\alpha_1 & -\alpha_2 & \ldots\ldots\ldots\ldots & -\alpha_n \end{bmatrix} \quad \text{----------} \qquad (2\text{-}14)$$

where $\alpha_1, \alpha_2, \dots \alpha_n$ are the known coefficients of the closed-loop characteristic equation

$$s^n + \alpha_n s^{n-1} + \alpha_{n-1} s^{n-2} + \dots + \alpha_2 s + \alpha_1 = 0 \quad \text{--- (2-15)}$$

We can rewrite (2-9) in terms of the transformed equation as

$$S^i_{j\ell} = \frac{1}{g'(s_i)} \operatorname{tr}\left[L(s_i)\frac{\partial A_o}{\partial a_{j\ell}}\right] \quad \text{------------------ (2-16)}$$

where

$L(s_i)$ is the adjoint$(s_i I - A_o)$, noting this is independent of q by the definition of A_o in (2-14).

Now substituting (2-13) into (2-16) we have

$$S^i_{j\ell} = \frac{1}{g'(s_i)} \operatorname{tr}\left[L(s_i)\frac{\partial}{\partial a_{j\ell}}\left(P(q)(A + b_q k)P(q)^{-1}\right)\right]$$

$$S^i_{j\ell} = \frac{1}{g'(s_i)} \operatorname{tr}\left[P(q)^{-1}L(s_i)P(q)\frac{\partial}{\partial a_{j\ell}}(A + b_q k)\right] \quad \text{-- (2-17)}$$

Since only the elements of A are varying

$$\frac{\partial}{\partial a_{j\ell}}(A + b_q k) = \frac{\partial A}{\partial a_{j\ell}} + \frac{\partial}{\partial a_{j\ell}}(b_q k)$$

that is

$$\frac{\partial}{\partial a_{j\ell}}(A + b_q k) = \frac{\partial A}{\partial a_{j\ell}} \quad \text{------------------------- (2-18)}$$

then using equation (2-18) in (2-17), one obtains for the sensitivity

$$S^i_{j\ell} = \frac{1}{g'(s_i)} P_\ell(q)^{-1} L(s_i) P_j(q) \quad \text{----------------} \quad (2\text{-}19)$$

where

$P_\ell(q)^{-1}$ is the ℓth row of $P(q)^{-1}$

$P_j(q)$ is the jth column of $P(q)$

Using (2-19) in equation (2-10) we have the sensitivity functional, J strictly as a function of the variable q. We can now minimize the functional J with respect to q; the required k necessary for the desired pole locations can then be easily determined by use of equation (2-13). Once k is known, the minimum sensitivity controller K is obtained by equation (2-8).

The above procedure is utilized throughout the remainder of this thesis to obtain the minimum sensitivity controller.

2.3 An Example

In this section a numerical example is presented to illustrate the minimum sensitivity design procedure described in the previous section.

Consider a linear time-invarient second-order system with two inputs and two outputs described by

$$\overset{\circ}{x} = Ax + Bu \quad \text{---------------------------------} \quad (2\text{-}20)$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Now using the unity-rank feedback matrix as defined by (2-5) and (2-8), equation (2-20) becomes

$$\left.\begin{aligned} \overset{\circ}{x} &= (A + b_q k)x + BR \\ \text{or} \qquad \overset{\circ}{x} &= \hat{A}x + BR \end{aligned}\right\} \qquad \text{(2-21)}$$

Since there are only two inputs we can write q in the following form

$$q = [1 \quad q_1]^T \qquad \text{(2-22)}$$

where q_1 the scalar is to be determined such that the sensitivity functional J is a minimum.

It can be shown that the matrix which transforms the pair $(\hat{A}, b_q)$ is the same as that which transforms the matrix pair (A, b_q) to phase-variable or canonical form [36].

To determine the transformation matrix, P(q), the controllability matrix Q is formed where

$$Q = [b_q \mid Ab_q] \qquad \text{(2-23)}$$

or

$$Q = \begin{bmatrix} q_1 & 2q_1 + 1 \\ q_1 + 1 & 2q_1 + 1 \end{bmatrix}$$

We then have

$$Q^{-1} = \begin{bmatrix} -1 & -1 \\ \dfrac{q_1 + 1}{2q_1 + 1} & \dfrac{q_1}{2q_1 + 1} \end{bmatrix}$$

The trarsformation matrix

$$P(q) = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

is obtained as follows:

i) set the last row of $Q^{-1} = p_1$

ii) solve the equation $p_2 = p_1 A$.

Therefore

$$p_1 = \begin{bmatrix} \frac{q_1 + 1}{2q_1 + 1} & -\frac{q_1}{2q_1 + 1} \end{bmatrix}$$

$$p_2 = \begin{bmatrix} \frac{1}{2q_1 + 1} & \frac{1}{2q_1 + 1} \end{bmatrix}$$

$$P(q) = \begin{bmatrix} \frac{q_1 + 1}{2q_1 + 1} & -\frac{q_1}{2q_1 + 1} \\ \frac{1}{2q_1 + 1} & \frac{1}{2q_1 + 1} \end{bmatrix}$$

ard

$$P(q)^{-1} = \begin{bmatrix} 1 & q_1 \\ -1 & q_1 + 1 \end{bmatrix}$$

The poles of the open-loop system are 0 and 2 as deiermined by the roots of $\det(sI - A) = 0$, which results in the open-loop equation

$$s(s - 2) = 0 \quad \text{----------------------------------} \quad (2\text{-}24)$$

Since the open-loop system is unstable, we wish to stabilize the system by using a constant gain feedback controller which will be

designed using pole assignment and minimum eigenvalue sensitivity. In order to obtain a "good" transient response we will choose new closed-loop system pole locations as $s_1 = -1$ and $s_2 = -2$. The characteristic equation of the closed-loop system is then

$$\left.\begin{aligned} (s+1)(s+2) &= 0 \\ s^2 + 3s + 2 &= 0 \end{aligned}\right\} \text{------------------------} \quad (2\text{-}25)$$

Comparing coefficients of equation (2-25) and (2-15), we get

$$\left.\begin{aligned} \alpha_1 &= 2 \\ \alpha_2 &= 3 \end{aligned}\right\} \text{--------------------------------} \quad (2\text{-}26)$$

In canonical form the closed-loop system matrix then has the form

$$Ao = \begin{bmatrix} 0 & 1 \\ -\alpha_1 & -\alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \text{-------------} \quad (2\text{-}27)$$

Now

$$L(s_i) = \operatorname{adj}(s_i I - Ao) = \begin{bmatrix} s_i + 3 & 1 \\ -2 & s_i \end{bmatrix}, \quad i = 1,2$$

for $i = 1$; $s_i = s_1 = -1$, we get

$$L(s_1) = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \quad \text{--------------------------} \quad (2\text{-}28)$$

and for $i = 2$; $s_i = s_2 = -2$, we get

$$L(s_2) = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \quad \text{--------------------------} \quad (2\text{-}29)$$

Now $g'(s_i)$ is the derivative of the closed-loop characteristic equation (2-25) evaluated at s_i $(i = 1,2)$, therefore

$$g'(s_i) = 2s_i + 3 \quad \text{----------------------------} \quad (2\text{-}30)$$

Thus for i = 1

$$g'(s_1) = 1 \qquad \text{(2-31)}$$

and for i = 2

$$g'(s_2) = -1 \qquad \text{(2-32)}$$

The sensitivity matrices can now be calculated using equation (2-19)

$$S^i_{j\ell} = \frac{1}{g'(s_i)} P_\ell(q)^{-1} L(s_i) P_j(q) \qquad \text{(2-33)}$$

$$\text{for } i = 1,2 \quad j = 1,2 \quad \ell = 1,2$$

then for i = 1, using the results of $P(q)$, $P(q)^{-1}$, (2-28) and (2-31)

$$S^1_A = \begin{bmatrix} -\frac{(2q_1+3)(q_1-1)}{2q_1+1} & -\frac{(2q_1+3)(q_1+2)}{2q_1+1} \\ \frac{(2q_1-1)(q_1-1)}{2q_1+1} & \frac{(2q_1-1)(q_1+2)}{2q_1+1} \end{bmatrix}$$

and for i = 2, using the results of $P(q)$, $P(q)^{-1}$, (2-29) and (2-32)

$$S^2_A = \begin{bmatrix} \frac{(2q_1-1)(q_1+2)}{2q_1+1} & \frac{(2q_1+3)(q_1+2)}{2q_1+1} \\ -\frac{(2q_1-1)(q_1-1)}{2q_1+1} & -\frac{(2q_1+3)(q_1-1)}{2q_1+1} \end{bmatrix}$$

Next we shall consider two cases of plant parameter variations. These are manifested as variations of the elements of the system matrix A.

Case 1

Denote the matrix A as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and assume the element a_{21} alone is subjected to variations. Then the sensitivity functional to be minimized is

$$J = \sum_{i=1}^{2} (|S_{21}^{i}|)^2$$

$$J = (|S_{21}^{1}|)^2 + (|S_{21}^{2}|)^2$$

$$J = \frac{2(2q_1 - 1)^2(q_1 - 1)^2}{(2q_1 + 1)^2} \quad \text{--------------------} \quad (2\text{-}34)$$

By inspection the minimum value of $J = 0$ occurs if $q_1 = \frac{1}{2}$ or 1. Infinite values of q_1 are disreguarded for practical reasons. Choosing $q_1 = 1$ gives the following results:

$$q = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \quad ; \quad b_q = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad ; \quad P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Now it is known that by using the transformation matrix P, the canonical form of $\hat{A}$ is Ao. Therefore using the relationship shown in equation (2-13) we can solve for the only unknown value, k, which assigns the poles to

the desired locations in the complex frequency plane. Thus we have

$$k = \begin{bmatrix} -3 & -1 \end{bmatrix} \qquad \text{(2-35)}$$

and using equation (2-8) we find the minimum sensitivity controller K to be

$$K = \begin{bmatrix} -3 & -1 \\ -3 & -1 \end{bmatrix} \qquad \text{(2-36)}$$

This feedback matrix, K, should result in the poles of the closed-loop system to be at the desired locations of $s_1 = -1$, $s_2 = -2$. This can easily be varified by taking $\det(sI - (A + BK)) = 0$.

Now let the element a_{21} vary such that $\Delta a_{21} = 0.2$, then

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix}$$

Denote

$$\bar{A} = A + \Delta A \qquad \text{(2-37)}$$

or

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 1.2 & 1 \end{bmatrix}$$

Then the perturbed closed-loop system matrix is

$$\tilde{A} = \bar{A} + BK \qquad \text{(2-38)}$$

or

$$\tilde{A} = \begin{bmatrix} -2 & 0 \\ -4.8 & -1 \end{bmatrix}$$

which results in a closed-loop characteristic equation of

$$s^2 + 3s + 2 = 0$$

$$(s + 2)(s + 1) = 0 \qquad \text{(2-39)}$$

indicating we have closed-loop pole locations of $s_1 = -1$ and $s_2 = -2$. Thus we see there are no variations in the pole locations when the element a_{21} varies from 1 to 1.2. In fact, as can be seen from $\tilde{A}$ any variation in a_{21} will have "no" affect on the pole locations.

Now suppose we disregard the sensitivity criterion and assign the closed-loop poles to the same locations as before by choosing an arbitrary q. Let $q = [1 \quad 10]^T$. Then following the procedure described earlier we get

$$K = \begin{bmatrix} -\frac{27}{21} & \frac{15}{21} \\ -\frac{270}{21} & \frac{150}{21} \end{bmatrix}$$

By taking $\det[sI - (A + BK)] = 0$, it is easily shown that the poles of the closed-loop system using arbitrary feedback are at the exact locations.

Now using the same variation in a_{21} as before one obtains for the closed-loop characteristic equation

$$s^2 + 3s + 0.3714 = 0 \qquad \text{(2-40)}$$

which has pole locations of $s_1 = -0.13$ and $s_2 = -2.87$. Thus we see that for a 20% change in a_{21}, s_1 changes by 87% and s_2 changes by 43.5%.*

---

* See Appendix III for the method used to calculate the percent change in pole locations.

Case 2

Now suppose all the elements of A are subjected to variations. The functional to be minimized is then

$$J = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{\ell=1}^{n} (|S_{j\ell}^{i}|)^2$$

$$J = (|S_{11}^{1}|)^2 + (|S_{12}^{1}|)^2 + (|S_{21}^{1}|)^2 + (|S_{22}^{1}|)^2$$

$$+ (|S_{11}^{2}|)^2 + (|S_{12}^{2}|)^2 + (|S_{21}^{2}|)^2 + (|S_{22}^{2}|)^2$$

and using the values obtained for S_A^1 and S_A^2 the above becomes

$$J = \frac{2}{(2q_1 + 1)^2}(16q_1^4 + 32q_1^3 + 76q_1^2 + 48q_1 + 50) \quad \text{-- (2-41)}$$

Equation (2-41) was solved on the digital computer at the University of Alberta center using the minimization Subroutine ZXMIN (Int. Mathematical Statistical Lib., Vol. 1, Ed. 5) to determine the minimum valve of J. The minimum value of J = 47.097 occurs when $q_1 = 0.714$. Then using the value of $q = \begin{bmatrix} 1 & 0.714 \end{bmatrix}^T$ the minimum sensitivity controller was determined as

$$K = \begin{bmatrix} -3.471 & -1.472 \\ -2.478 & -1.05 \end{bmatrix}$$

This feedback matrix obtained using the minimum sensitivity design assigns the poles of the closed-loop system to the exact locations of $s_1 = -1$ and $s_2 = -2$.

Taking the variation in A to be

$$\Delta A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & 0.1 \end{bmatrix} \quad \text{-------------------------} \quad (2\text{-}45)$$

results in the perturbed closed-loop system matrix of

$$\tilde{A} = \begin{bmatrix} -1.378 & -0.15 \\ -4.849 & -1.422 \end{bmatrix}$$

which has pole locations of

$$s_1 = -0.55$$
$$s_2 = -2.25$$

Thus we see that s_1 changes by 45% and s_2 changes by 12.5%.

Now using an arbitrary controller by choosing $q = [1 \quad 10]^T$ one obtains (as before) the feedback matrix of

$$K = \begin{bmatrix} -\frac{27}{21} & \frac{15}{21} \\ -\frac{270}{21} & \frac{150}{21} \end{bmatrix}$$

Then subjecting A to the same variations as shown in (2-42) and using the above K, one obtains the poles of the perturbed closed-loop system as

$$s_1 = 0.14$$
$$s_2 = -2.94$$

indicating that s_1 changes by 114% and s_2 changes by 47%. Also as can be seen by the above pole locations, the arbitrary system becomes unstable when A is subjected to the variation indicated.

Thus one can see that the minimum sensitivity controller design has definite advantages over the arbitrary feedback controller design for pole assignment.

Both systems were simulated with the external control input $R = [1.0 \quad 1.0]^T$ for the cases $\Delta A = 0$ and $\Delta A =$ equation(2-42). The responses obtained when using the minimum sensitivity controller are shown in figure 2 for x_1 and in figure 4 for x_2; when using the arbitrary controller the responses are shown in figure 3 for x_1 and figure 5 for x_2.

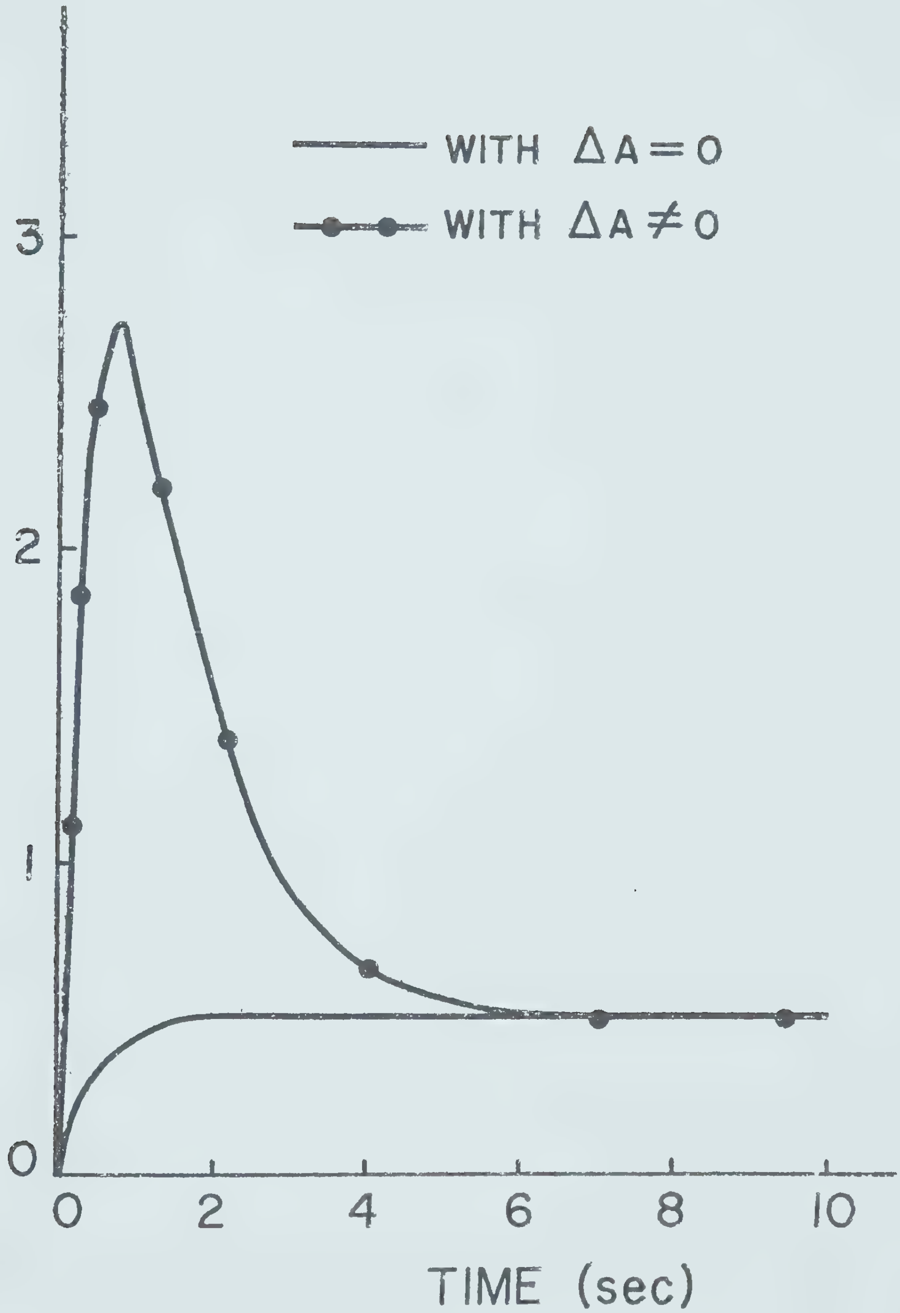

Figure 2. Response of X_1 Using the Minimum Sensitivity Controller

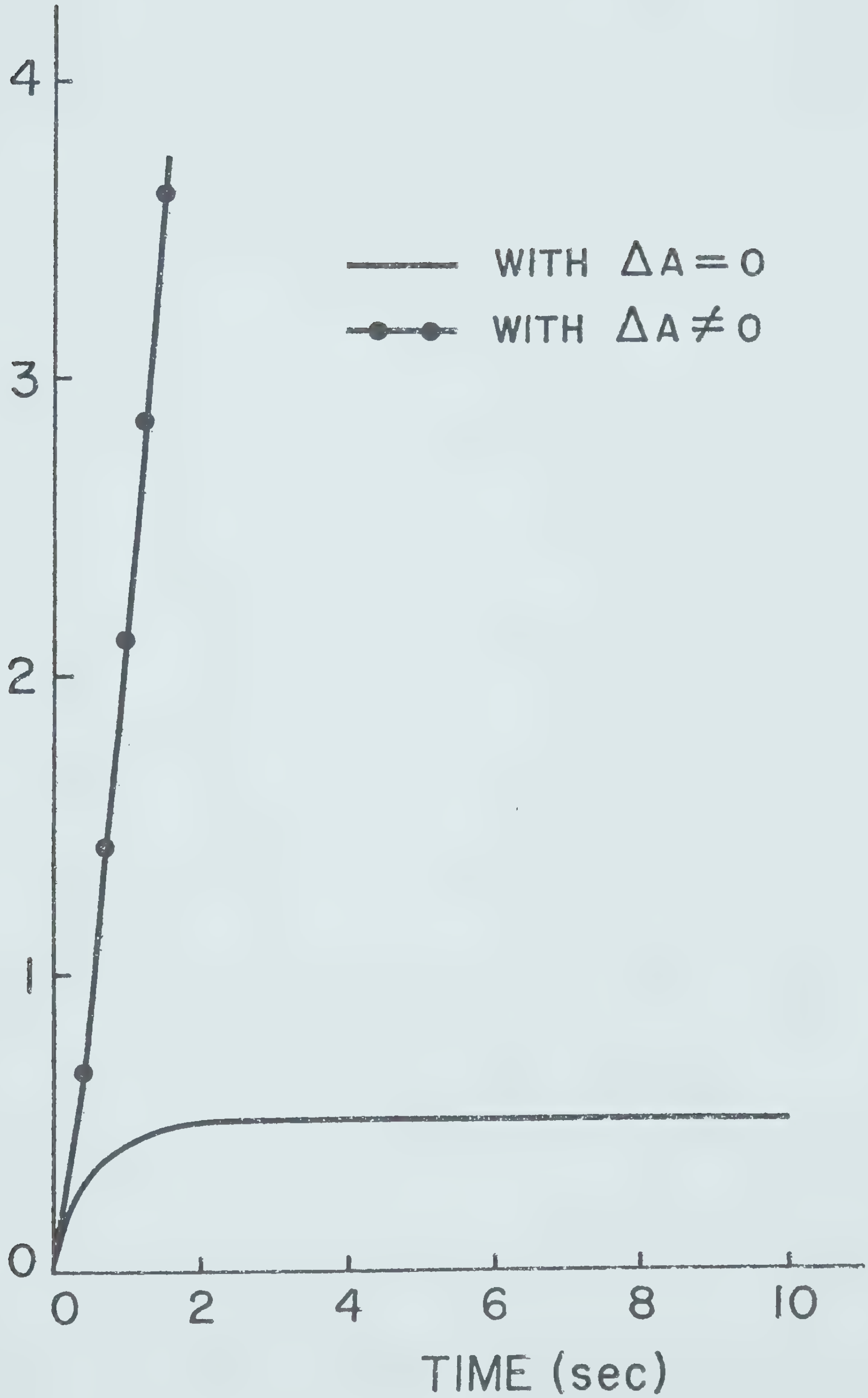

Figure 3. Response of X_1 Using the Arbitrary Controller

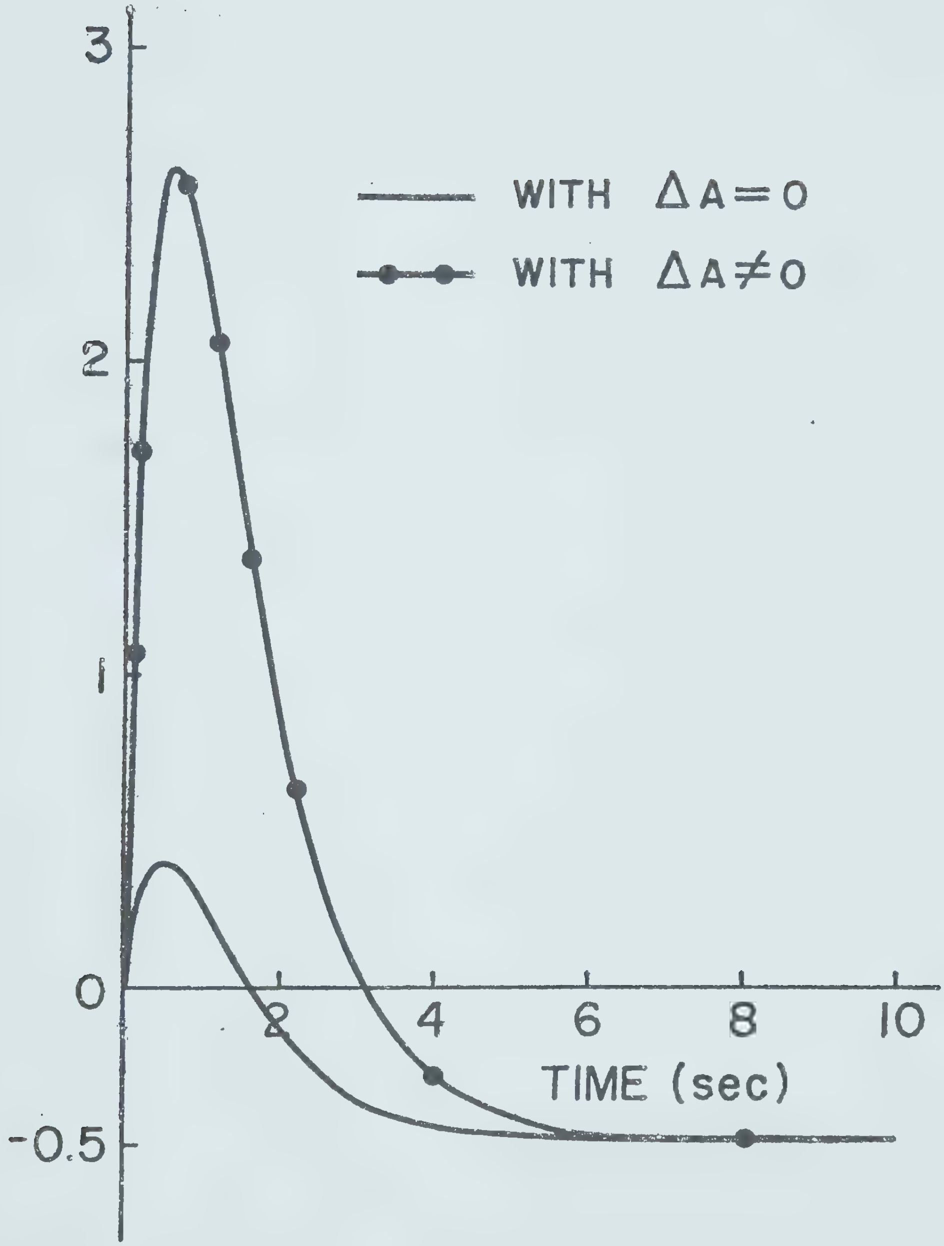

Figure 4. Response of X_2 Using the Minimum Sensitivity Controller

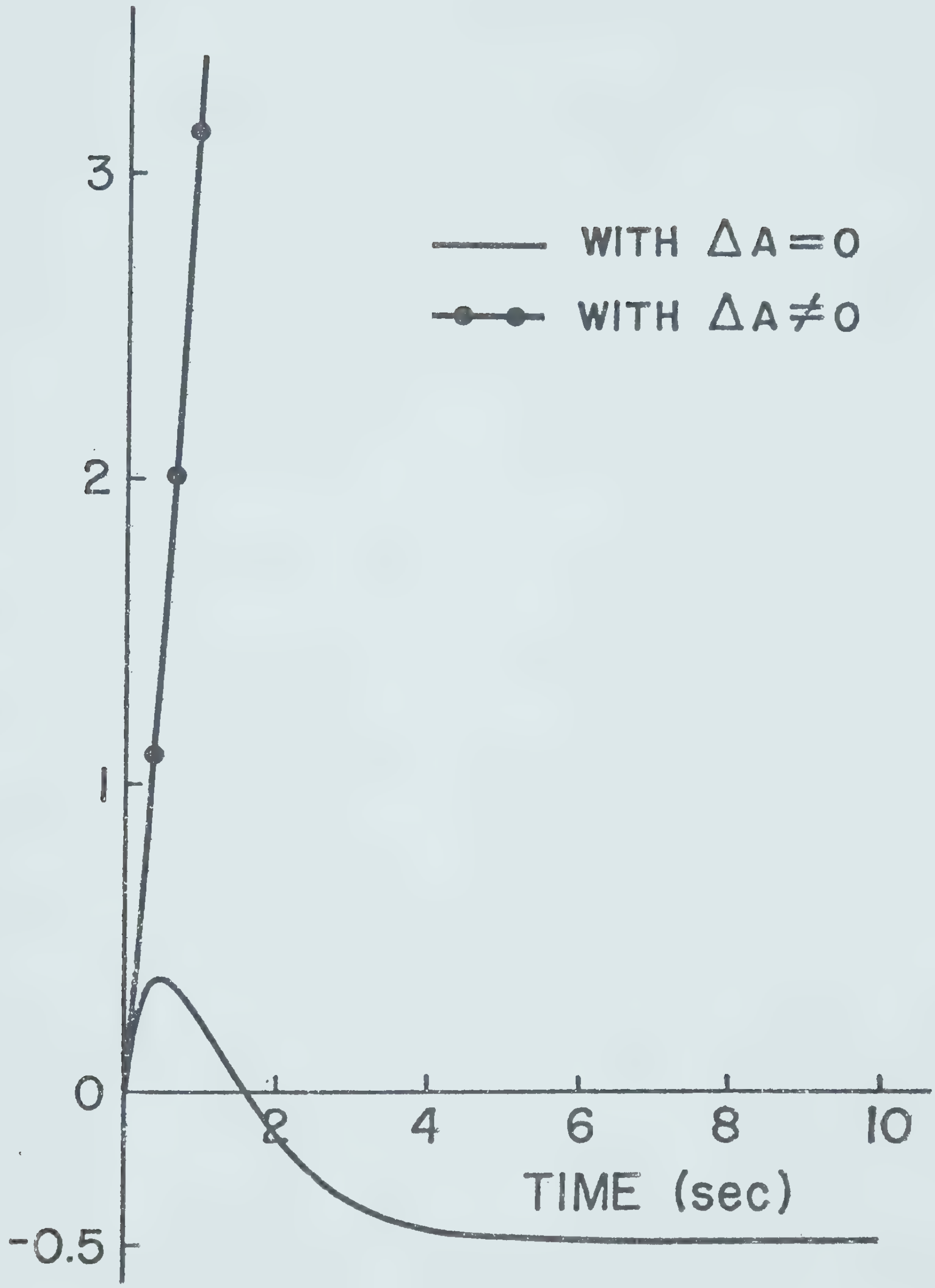

Figure 5. Response of X_2 Using the Arbitrary Controller

CHAPTER (3)

VTOL CONTROLLER DESIGN

3.1 General objectives of Feedback Controller

In general, the dynamics of a VTOL aircraft are such that it is inherently unstable. For this aircraft to be of any practical use, some form of stability and/or control augmentation is required so that it will be stable and controllable during its flight regime. The introduction of feedback controllers into the VTOL system is one way of achieving this.

Over the past fifty years or so many controllers have been designed and proven to be useful on existing VTOL aircraft. Some of these are outlined in the references at the end of this thesis.

Although stability is perhaps the main criterion in the design of a VTOL aircraft controller it is not by far the only factor to be considered. Response of the system to predetermined or disturbance inputs, cost of implementing the control system and reliability of the system are also important criteria to be considered when designing a controller.

It is shown in this thesis (in this chapter and the next) that a constant gain feedback controller designed by using the minimum eigenvalue or pole assignment enhances the stability and transient response characteristics of the VTOL aircraft considered, even under conditions of plant parameter variations. Also the cost of implementing a constant feedback control scheme (as presented here) is relatively small compared to some of the other more sophisticated control schemes, such as the adaptive control scheme proposed by Narendra and Tripathi [1] .

3.2 Model Dynamics

The model used in the design procedure is that used by K.S. Narendra and S.S. Tripathi [1]. As in most other techniques used for VTOL controller design this model is obtained by linearization of the system dynamics around a nominal air speed.

The linearized model of the VTOL aircraft in the vertical plane is described by

$$\overset{\circ}{x} = Ax + Bu \qquad \text{------------------------------------} \quad (3\text{-}1)$$

where A is the (4x4) system matrix, B is the (4x2) control matrix, x is the 4-dimensional state vector and u is the 2-dimensional control vector. The state variables are:

x_1 - horizontal velocity;

x_2 - vertical velocity;

x_3 - pitch rate, and

x_4 - pitch angle.

The control inputs are:

u_1 - collective, and *

u_2 - longitudinal cyclic.

---

* The control u_1 is located on the collective pitch lever at the pilots side. Its main use is the control over the vertical velocity of the VTOL aircraft by the selection of a desired flight angle. The flight path or angle is changed merely by the pilot applying collective control pressure in an up or down motion. This control also has some affect on the horizontal velocity.

The control u_2 is one of the controls located on the cyclic control stick immediately in front of the pilot. Its main use is to control the horizontal velocity of the VTOL aircraft. The up and down movement of the control (the one we are considering) changes the forward horizontal velocity of the helicopter while a left or right movement of the control will change the heading reference to that of the flight indicator.

In this thesis, following [1] , the nominal airspeed is assumed to be one hundred and thirty-five (135) knots. At this nominal airspeed the system matrix A and control matrix B are:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.707 & 1.42 \\ 0.0 & 0.0 & 1.0 & 0.0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0.0 & 0.0 \end{bmatrix}$$

As the aircraft speed deviates from the nominal air speed all the elements in the first three rows of both matrices change. The most significant changes occuring in the elements a_{32} and a_{34}, the rest of the elements can be assumed to remain constant without serious loss of accuracy.*

The design technique described in chapter (2) is used here to take into account the changes in the parameters a_{32} and a_{34} as the airspeed changes from sixty (60) to one hundred and seventy (170) knots.

---

* In this thesis the design technique is used only to take into account the changes in the elements of the plant matrix A. However, if found necessary, this design can easily be modified to incorporate the changes in the control matrix B as well [18] .

It is assumed that the elements a_{32} and a_{34} vary in a linear manner as the airspeed changes from sixty to one hundred and thirty-five knots, and from one hundred and thirty-five to one hundred and seventy knots. Although the above assumption is not entirely valid, it is acceptable to illustrate the design procedure. The manner in which a_{32} and a_{34} vary as a function of time is shown in figures 6 and 7 respectively. Note that the time shown in figures 6 and 7 is only approximate as it had to be taken from the graphs of reference 1, Page 194.

Since two elements of the matrix A are changing with changes in airspeed the functional to be minimized is

$$J = \sum_{i=1}^{n} (|S_{32}^{i}|)^2 + (|S_{34}^{i}|)^2 \quad \text{------------------} \quad (3\text{-}2)$$

Since the closed-loop system is of 4th order, there are four eigenvalues. The desired locations for these eigenvalues are

$$s_1 = -1.5$$
$$s_2 = -2.0$$
$$s_3 = -1 + j1$$
$$s_4 = -1 - j1$$

The Fortran program listed in Appendix I was used to carry out necessary computations for the entire design procedure. This program was written to incorporate the Subroutine ZXMIN which computes the minimum of the junctional J (3-2).

The minimum value of $J = 0.0054$ was obtained when $q_1 = 0.677$. This resulted in the minimum sensitivity feedback controller

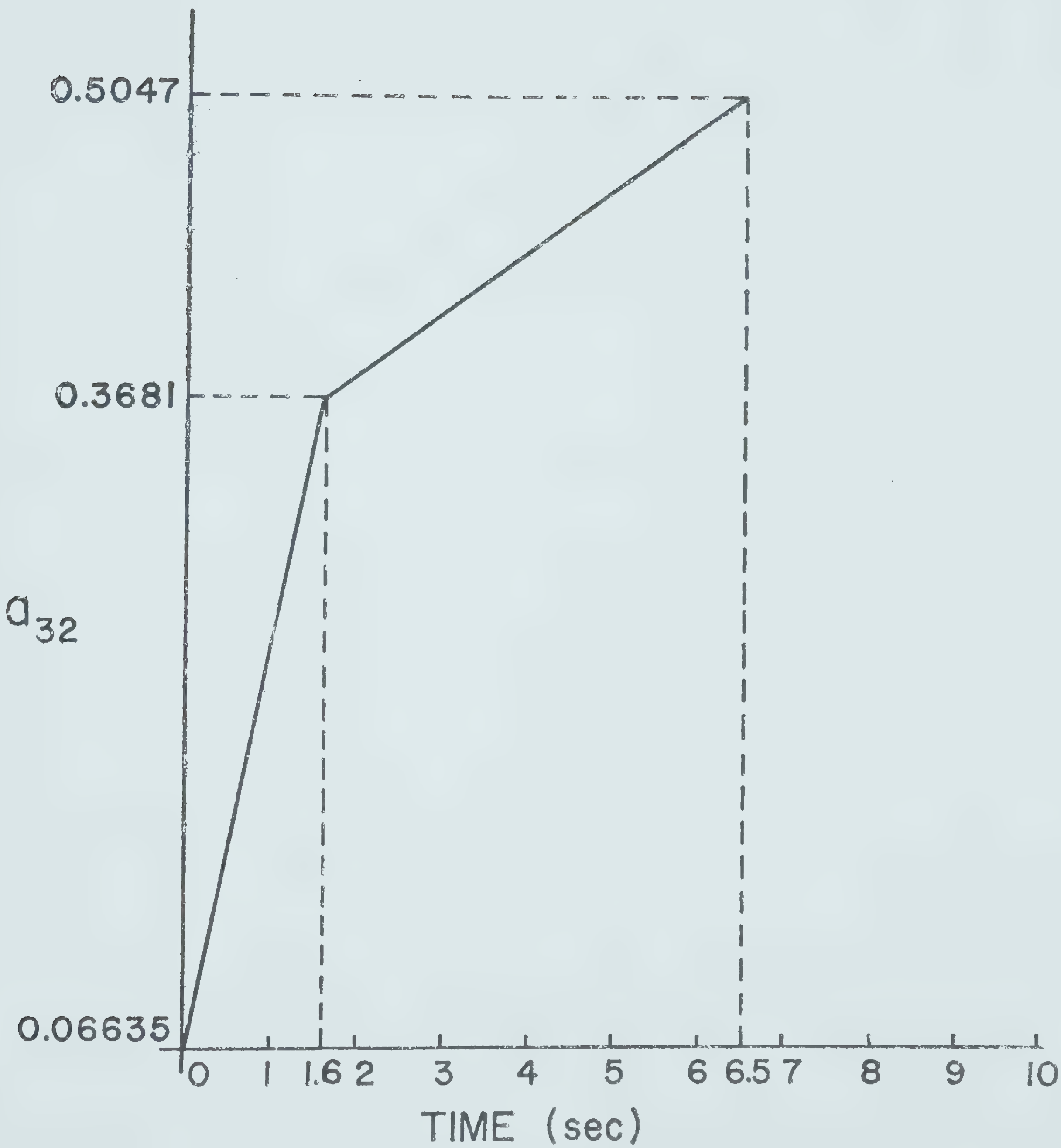

Figure 6. Change in the Parameter a_{32} of the System Matrix A as a function of Time

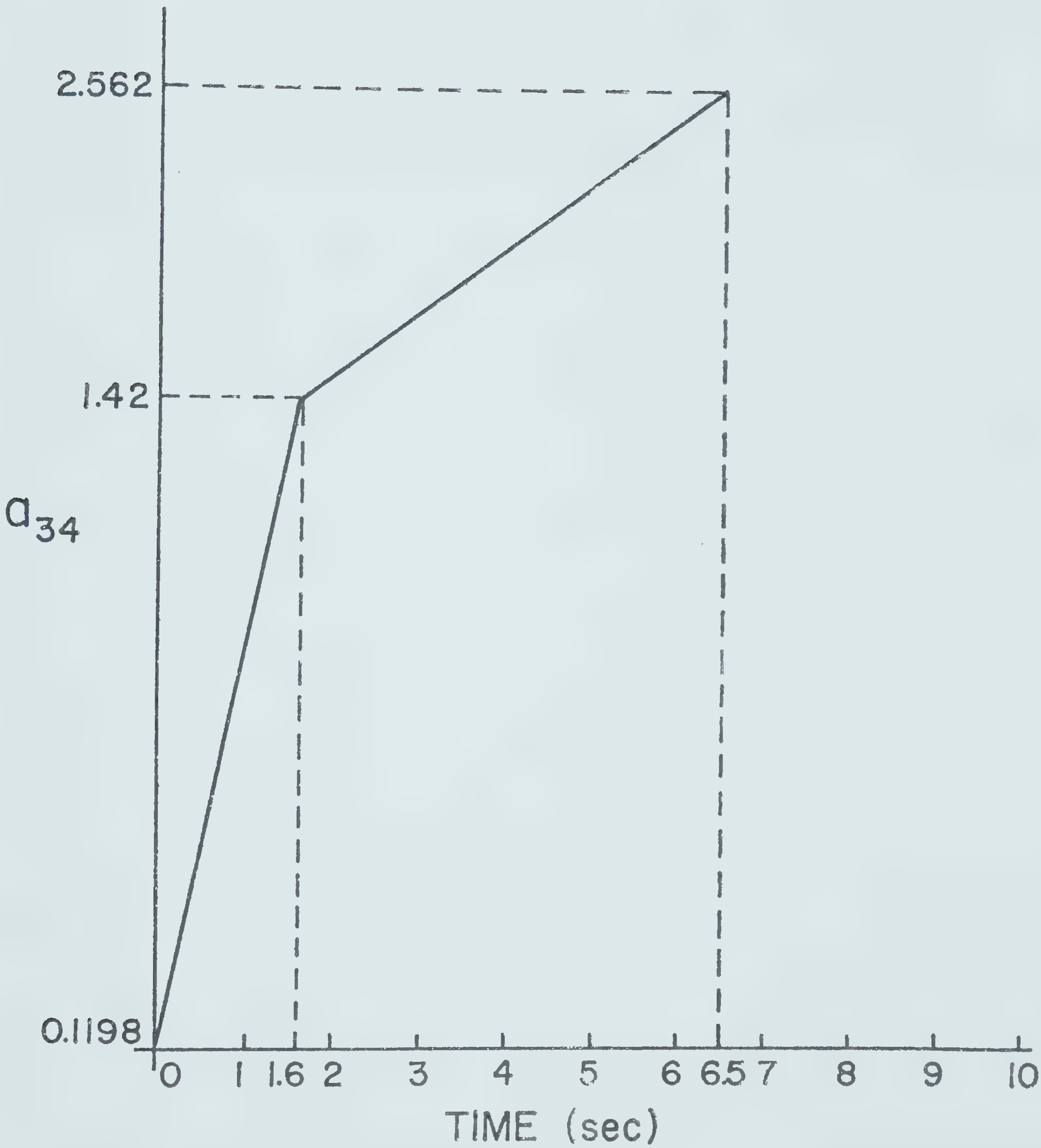

Figure 7. Change in the Parameter a_{34} of the System Matrix A as a Function of Time

$$K = \begin{bmatrix} -3.268 & -0.1334 & 0.8567 & 2.930 \\ -2.212 & -0.0903 & 0.5799 & 1.984 \end{bmatrix} \quad \text{--- (3-3)}$$

At the nominal airspeed of one hundred and thirty-five knots, this feedback controller (3-3), assigns the closed-loop poles exactly to the desired locations.

Now using the minimum sensitivity controller (3-3) the system was cheked for stability and pole variation of the closed-loop system at the extremes of the flight regime speeds, i.e. at sixty and at one hundred and seventy knots. At sixty knots $a_{32} = 0.06635$, $a_{34} = 0.1198$ and at one hundred and seventy knots $a_{32} = 0.5047$ and $a_{34} = 2.526$.

At sixty knots:

Denote

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta a_{32} & 0 & \Delta a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{------------------ (3-4)}$$

where

$$\Delta a_{32} = -0.30175$$

$$\Delta a_{34} = -1.3002$$

Let

$$\tilde{A} = A + \Delta A \quad \text{-------------------------------------- (3-5)}$$

Now replacing A by $\tilde{A}$ in equation (2-6) one obtains for the closed-loop system

$$\left.\begin{aligned} \mathring{x} &= (\tilde{A} + BK)x + BR \\ \mathring{x} &= \tilde{\hat{A}}x + BR \end{aligned}\right\} \qquad (3\text{-}6)$$

where $\tilde{\hat{A}}$ is the closed-loop system matrix, and R is the external control vector, the magnitude of which is determined by the pilot.

When the pole locations of $\tilde{\hat{A}}$ were checked it was found that

$$s_1 = -0.796$$
$$s_2 = -1.55$$
$$s_3 = -1.58 + j1.33$$
$$s_4 = -1.58 - j1.33$$

which indicates that the system is stable at the lower extreme of the flight regime. Thus one can see that for a $\Delta a_{32} = -0.30175$ which corresponds to a 82% change in a_{32} and for a $\Delta a_{34} = -1.3002$ which corresponds to a 92% change in a_{34} the closed-loop poles change by

$\Delta s_1 = 0.704$ or a 47% change in s_1

$\Delta s_2 = 0.45$ or a 22.5% change in s_2

$\Delta s_3 = -0.58 + j0.33$ or a 46% change in s_3 @ $\underline{/4.9^{\circ}}$

$\Delta s_4 = -0.58 - j0.33$ or a 46% change in s_4 @ $\underline{/-4.9^{\circ}}$

At one hundred and seventy knots:

Using the same notation as before

$$\Delta a_{32} = 0.1365$$
$$\Delta a_{34} = 1.106$$

Again checking the pole locations of $\tilde{\hat{A}}$ the closed-loop system resulted in pole locations of

$s_1 = -1.52$

$s_2 = -2.56$

$s_3 = -0.71 + j0.946$

$s_4 = -0.71 - j0.946$

Thus for a change in a_{32} of 37% and a change in a_{34} of 78% the closed-loop poles changed by

$\Delta s_1 = -0.02$ or a 1.3% change in s_1

$\Delta s_2 = -0.56$ or a 28% change in s_2

$\Delta s_3 = 0.29 - j0.054$ or a 20.8% change in s_3 @ $\underline{/-8^0}$

$\Delta s_4 = 0.29 + j0.054$ or a 20.8% change in s_4 @ $\underline{/8^0}$

Again we see that the system is stable at the upper end of the VTOL flight regime.

By using the standard method for pole assignment incorporating unity rank feedback one can obtain for the VTOL system by choosing arbitrarily, $q = \begin{bmatrix} 1 & 10 \end{bmatrix}^T$, the feedback controller

$$K = \begin{bmatrix} 1.063 & -0.302 & -0.709 & -0.535 \\ 10.63 & -3.02 & -7.09 & -5.35 \end{bmatrix} \qquad \text{--------} \quad (3\text{-}7)$$

At the nominal air speed, it can easily be shown the arbitrary controller of (3-7) assigns the closed-loop poles to the exact desired locations as in the previous instance.

At sixty knots:

The closed-loop poles were found to be

$s_1 = -2.92 + j4.65$

$s_2 = -2.92 - j4.65$

$s_3 = 0.167 + j0.764$

$s_4 = 0.167 - j0.764$

From the location of the poles of the closed-loop system it is clear that the system is unstable when the plant matrix A is subjected to the variation of $\Delta a_{32} = -0.30175$ and $\Delta a_{34} = -1.3002$ which resulted in pole location changes of

$$\Delta s_1 = -1.42 + j4.65 \text{ or a 121\% change in } s_1 \text{ @ } \underline{/-57.9}^{0}$$

$$\Delta s_2 = -0.92 - j4.65 \text{ or a 118.5\% change in } s_2 \text{ @ } \underline{/57.9}^{0}$$

$$\Delta s_3 = 1.167 - j0.236 \text{ or a 84\% change in } s_3 \text{ @ } \underline{/-57.3}^{0}$$

$$\Delta s_4 = 1.167 + j0.236 \text{ or a 84\% change in } s_4 \text{ @ } \underline{/\ 57.3}^{0}$$

for a change in a_{32} and a_{34} of 82% and 92% respectively.

At one hundred and seventy knots:

The closed-loop poles were found to be

$$s_1 = -0.026$$

$$s_2 = -5.83$$

$$s_3 = 0.176 + j2.4$$

$$s_4 = 0.176 - j2.4$$

Again by the location of the poles in the complex frequency plane one can see that the system is unstable when the plant matrix A is subjected to the variation of $\Delta a_{32} = 0.1365$ and $\Delta a_{34} = 1.106$ which resulted in pole location changes of

$$\Delta s_1 = 1.474 \quad \text{or a 98.3\% change in } s_1$$

$$\Delta s_2 = -3.83 \quad \text{or a 91.5\% change in } s_2$$

$$\Delta s_3 = 1.176 + j1.4 \text{ or a 129\% change in } s_3 \text{ @ } \underline{/-49.2}^{0}$$

$$\Delta s_4 = 1.176 - j1.4 \text{ or a 129\% change in } s_4 \text{ @ } \underline{/49.2}^{0}$$

for a change in a_{32} and a_{34} of 37% and 78% respectively.

Thus one can see that the design procedure which minimizes eigenvalue sensitivity has definite advantages over an arbitrary controller

which also assigns the closed-loop poles to the desired locations at the nominal speed of operation.

3.3 Tracking Capabilities of the Controller

The ability of a control system to alter the steady state condition of the aircraft to a new steady state condition in a satisfactory manner, as determined by pilot control inputs, is a measure of how well the system is able to track or follow desired command inputs. An over damped system response will result in the aircraft being too sluggish or slow to reach its new desired steady state condition where as an underdamped system may result in uncontrollable oscillations of the aircraft when it is subjected to command inputs. A schematic diagram of the VTOL control system in the tracking mode is shown in figure 8.

By choosing the dominant poles of the closed-loop system as

$$s_{3,4} = -1 \pm j1$$

at the nominal airspeed of one hundred and thirty-five knots results in a damping ratio of $\zeta = 0.707$ and the undamped natural frequency of oscillation $\omega_n = 1.414$ radians/second. At sixty knots the damping ratio $\zeta = 0.765$ and $\omega_n = 2.065$ radians/second, and at one hundred and seventy knots $\zeta = 0.6$ and $\omega_n = 1.183$ radians/second.

The above values of ζ and ω_n are within the boundries of $\zeta > 0.45$ of critical and 0.5 radians/second $< \omega_n <$ 5 radians/second as determined by Buffum and Robertson [10].

To check the tracking capabilities of the minimum sensitivity controller and the arbitrary controller the following procedure was adopted.

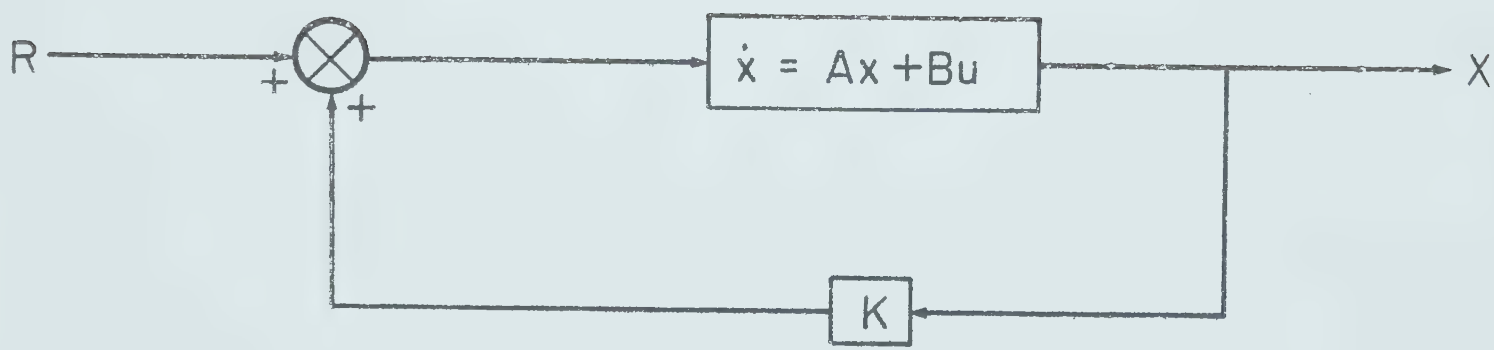

Figure 8. VTOL Control System: Tracking Operation

As mentioned previously, the model being used is that used by Narendra and Tripathi [1]. From [1] it is impossible to determine the exact pilot input which corresponds to any particular air speed. For this reason, in the following simulations, the pilot inputs (in this case R) were choosen as step inputs representing a particular horizontal velocity (see Table I). The pilot inputs are used to generate the corresponding control input to the system which results in an arbitrary steady state condition for each of the four output states x_1, x_2, x_3 and x_4 (the arbitrary steady states are obtained since we are arbitrarily selecting the pilot input to represent a particular air speed).

At sixty knots when $a_{32} = 0.06635$ and $a_{34} = 0.1196$, the pilot input was chosen to be $R = [0.4 \quad 0.4]^T$; at one hundred and thirty-five knots $a_{32} = 0.3681$ and $a_{34} = 1.42$ the pilot input was taken as $R = [1.0 \quad 1.0]^T$; and at one hundred and seventy knots $a_{32} = 0.5047$, $a_{34} = 2.526$ and the pilot input was taken as $R = [1.25 \quad 1.25]^T$.

The system was simulated on the digital computer using the CSMP program listed in Appendix II to determine the steady state values of the variables corresponding to each of the three steady state conditions mentioned in the previous paragraph.

It was now assumed that the aircraft was flying in steady state at sixty knots with $R = [0.4 \quad 0.4]^T$ and initial conditions on the variables as determined previously. Also by prior simulation at one hundred and thirty-five knots the steady state values of the variables at this air speed were also determined.

The pilot input was now suddenly increased from $R = [0.4 \quad 0.4]^T$ to $R = [1.0 \quad 1.0]^T$ to simulate an acceleration from sixty to one hundred

TABLE NO. I

Pilot Control Inputs Corresponding to a Certain Horizontal Velocity

PILOT CONTROL INPUTS (step inputs)		VTOL HORIZONTAL VELOCITY (knots)
r_1 collective	r_2 longitudinal cyclic	
0.4	0.4	60
1.0	1.0	135
1.25	1.25	170

and thirty-five knots. Since the parameters a_{32} and a_{34} change with changes in air speed, the change in the system matrix A was programed into the simulation (this was done by modifying the program in Appendix II). The response of the aircraft was then determined and plotted as shown in figures 9, 10, 11 and 12 for x_1, x_2, x_3 and x_4 respectively.

The above procedure was then repeated to simulate an acceleration from one hundred and thirty-five to one hundred and seventy knots. The response was again obtained and plotted as shown in figures 13, 14, 15 and 16 for x_1, x_2, x_3 and x_4 respectively.

Now for comparision purposes an arbitrary feed matrix was used (the one determined in Section 3.2). The previously determined steady state values of x_1, x_2, x_3 and x_4 at sixty, one hundred and thirty-five and one hundred and seventy knots respectively were used to determine the respective pilot input required to achieve the same steady state values at the desired air speeds when the arbitrary controller is used on the same aircraft. It is felt this is valid reasoning since for a completely different controller the pilot inputs will be different to achieve the same steady state results. Using this method it was found that to obtain the same steady state values as previous at sixty, one hundred and thirty-five and one hundred and seventy knots that $R = [-0.59 \quad -7.68]^T$, $R = [-1.25 \quad -13.9]^T$ and $R = [-1.45 \quad -15.15]^T$ respectively.

The system was then simulated starting in steady state at sixty knots ($R = [-0.59 \quad -7.68]^T$) and suddenly accelerating to one hundred and thirty-five knots (R changed to $R = [-1.25 \quad -13.9]^T$). The response was obtained, plotted and compared to the response obtained under the same conditions using the minimum sensitivity controller shown in

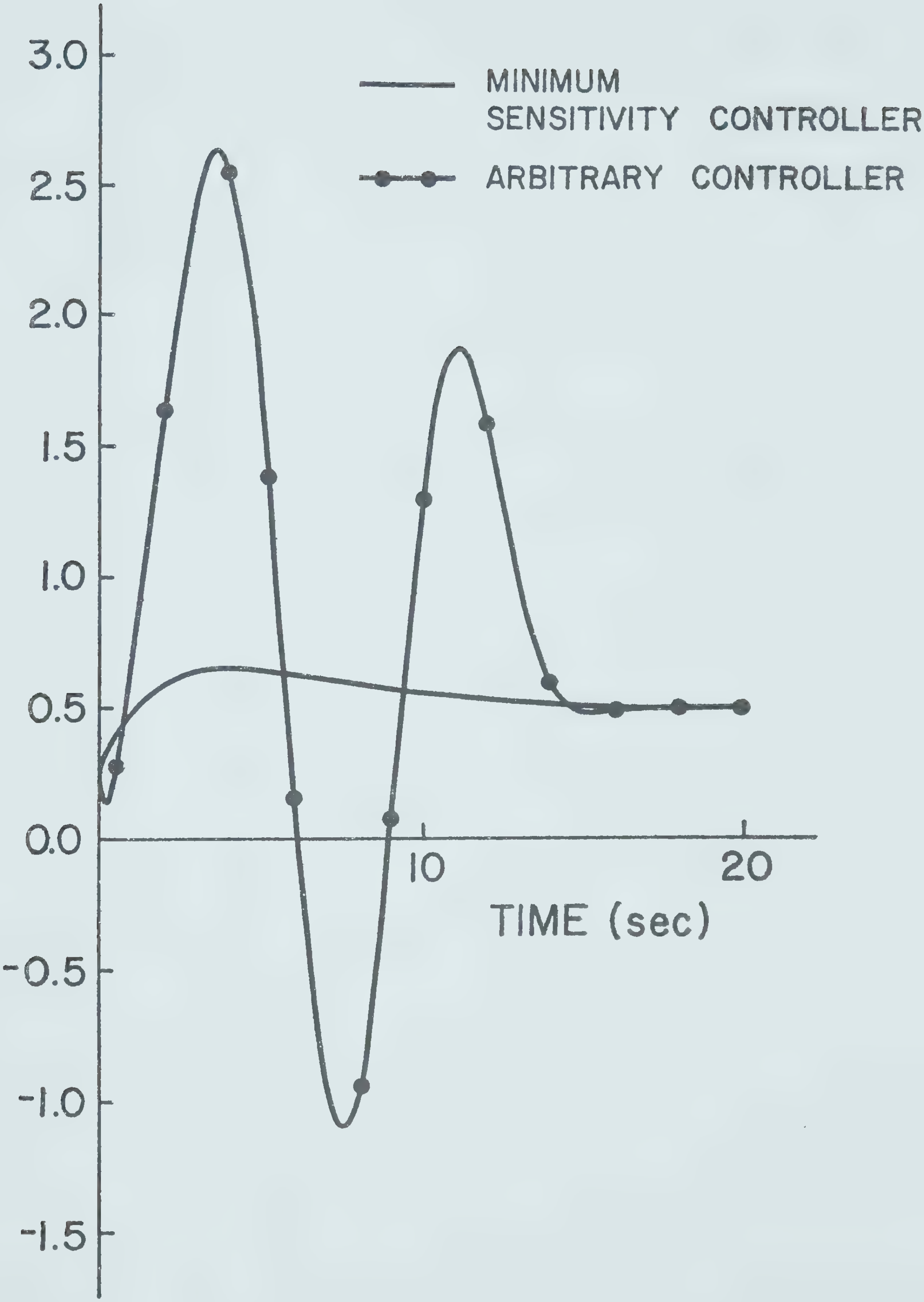

Figure 9. Response of X_1 when Accelerating from Sixty to One Hundred and Thirty-five Knots

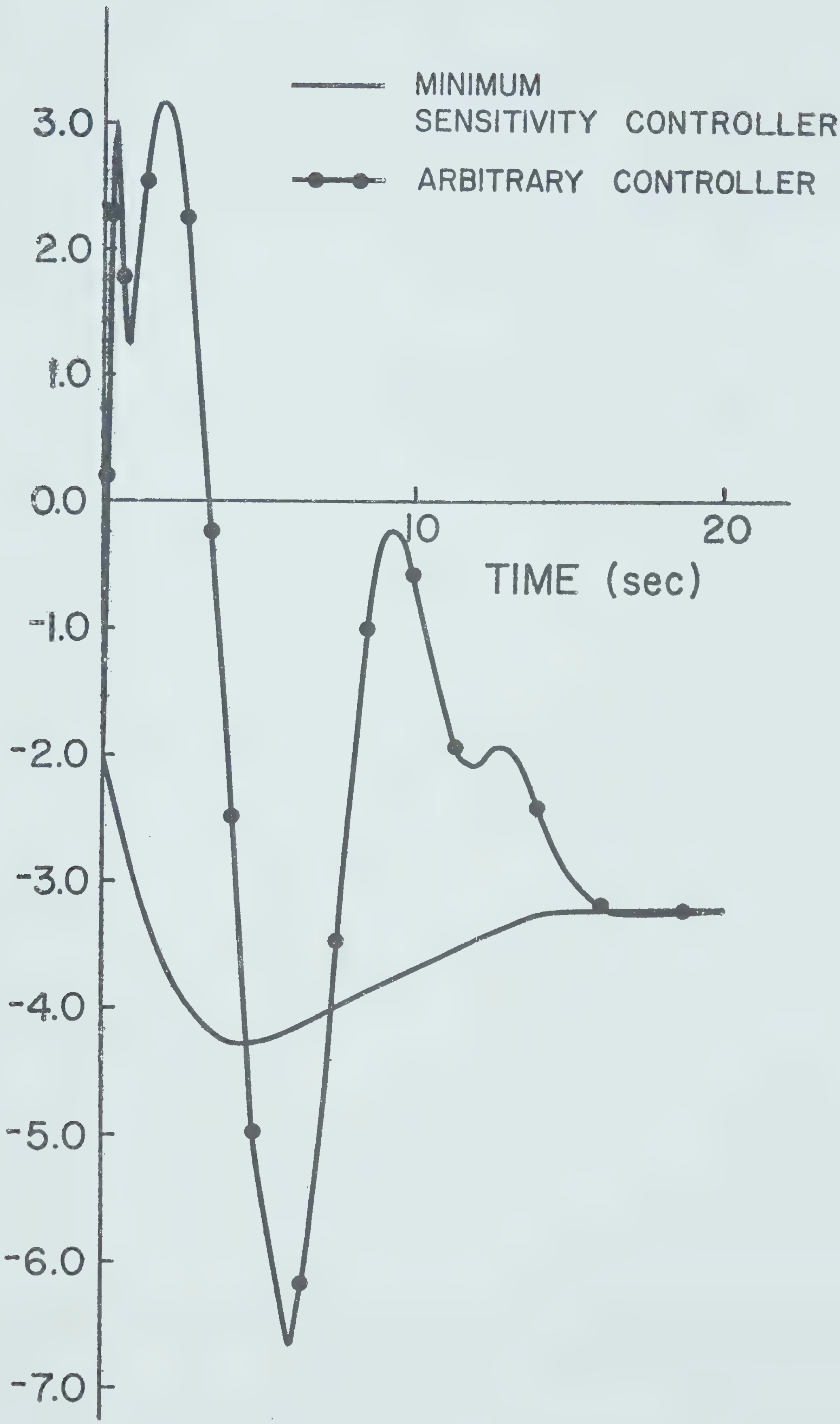

Figure 10. Response of X_2 when Accelerating from Sixty to One Hundred and Thirty-five Knots

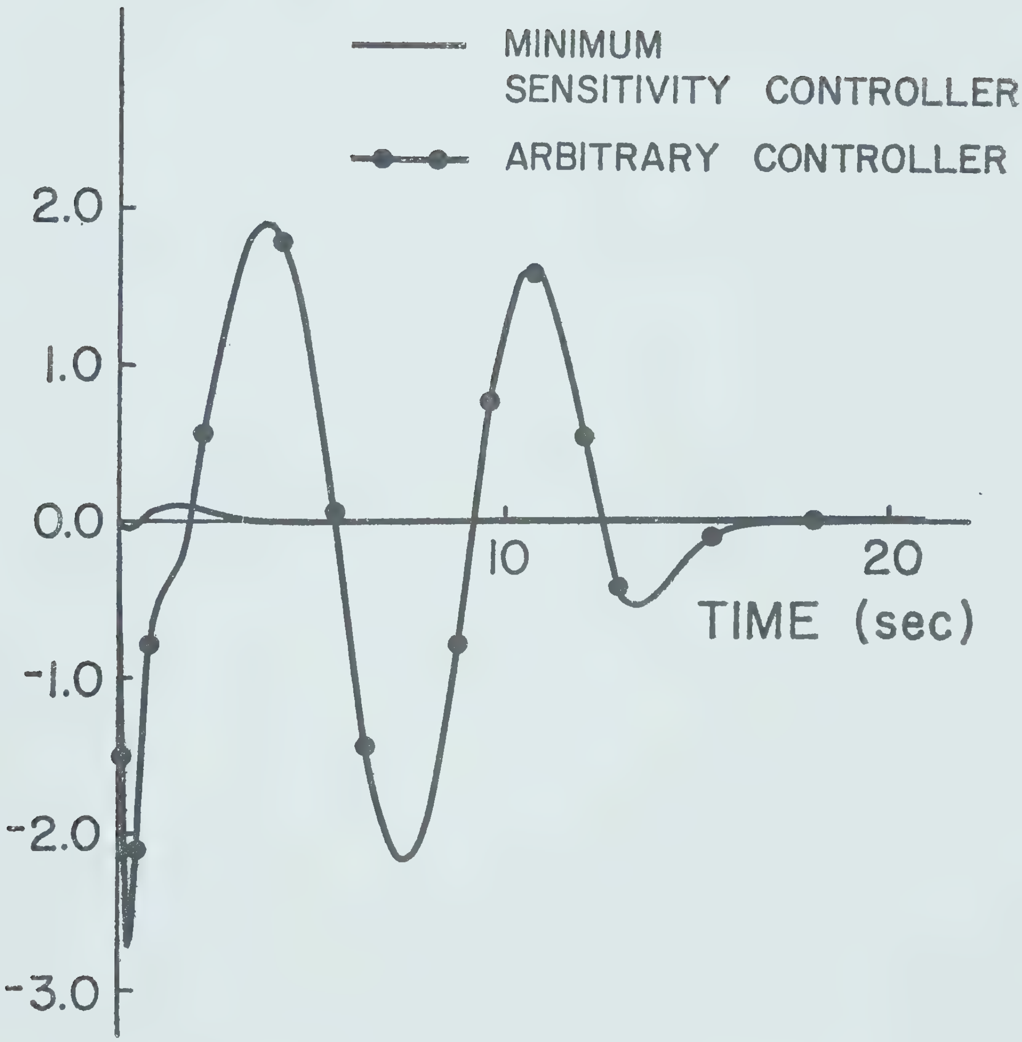

Figure 11. Response of X_3 when Accelerating from Sixty to One Hundred and Thirty-five Knots

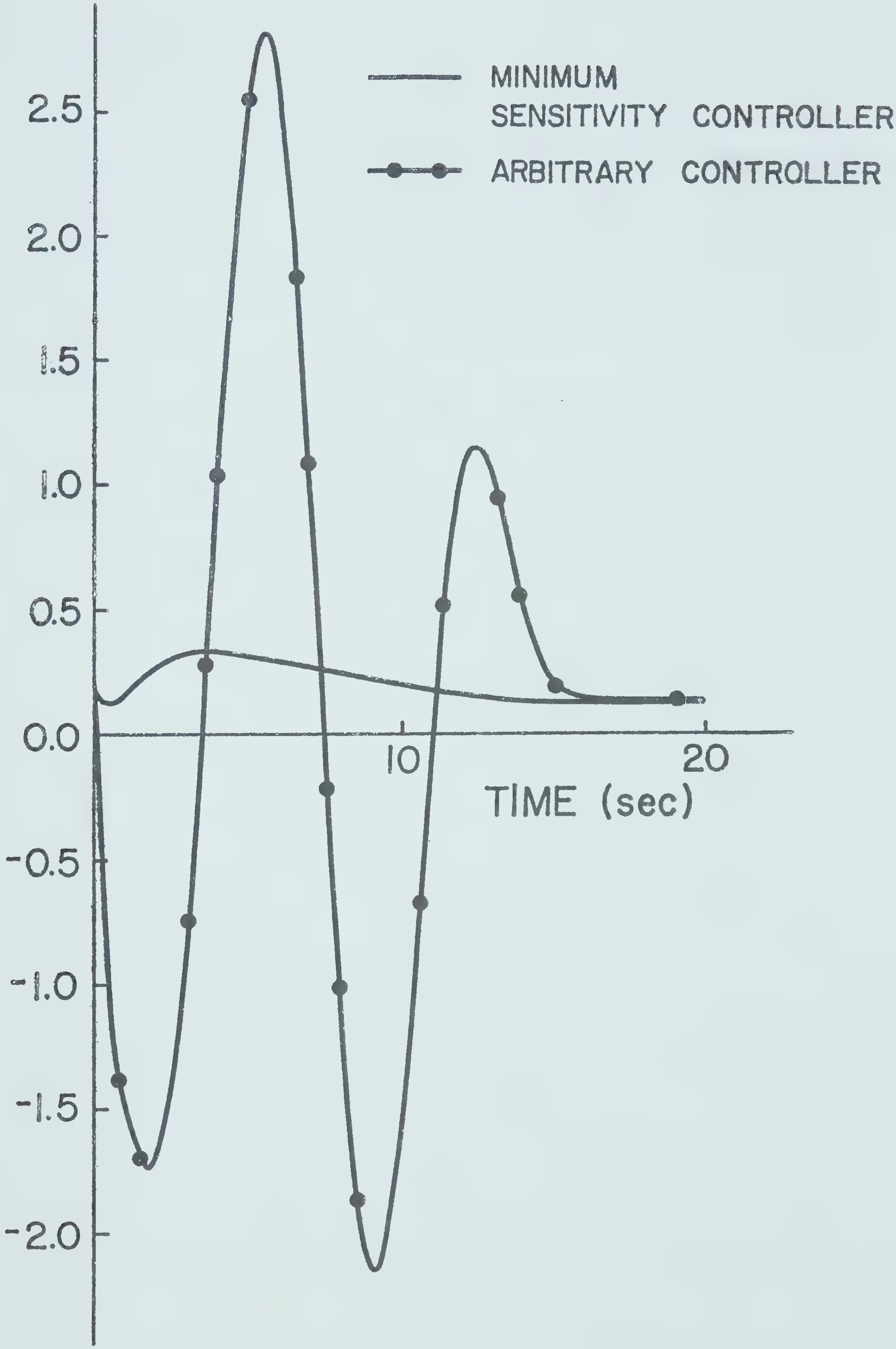

Figure 12. Response of X_4 when Accelerating from Sixty to One Hundred and Thirty-five Knots

figures 9, 10, 11 and 12.

The above procedure was then repeated to simulate an acceleration from one hundred and thirty-five to one hundred and seventy knots using the appropriate pilot input and initial conditions. The results obtained were then plotted and compared to those obtained for the same situation when using the minimum sensitivity controller, see figure 13, 14, 15 and 16.

It must be noted that although we are concerned in this design mainly with controlling the horizontal speed of the aircraft that the control inputs used, namely collective and longitudinal cyclic, produce changes in the other variables as well as the horizontal velocity variable x_1. This is common in all VTOL type aircraft as the modes of operation of the VTOL aircraft are not sufficiently decoupled so that changes in one mode of operation will not affect changes in another [2]. The longitudinal cyclic input from the control column is the main control over the horizontal speed of the aircraft. The collective input from the collective pitch lever affects not only the horizontal velocity but also the vertical velocity. Just how one affects the other is quite complicated and beyond the scope of this thesis. The references listed at the end of this thesis provide ample information on this aspect [15].

3.4 Regulating Capabilities of the Controller

The regulating ability of any controller is quite important, especially in a VTOL aircraft. By regulation is meant the ability of the controller to maintain a steady state condition in the presence of disturbances.

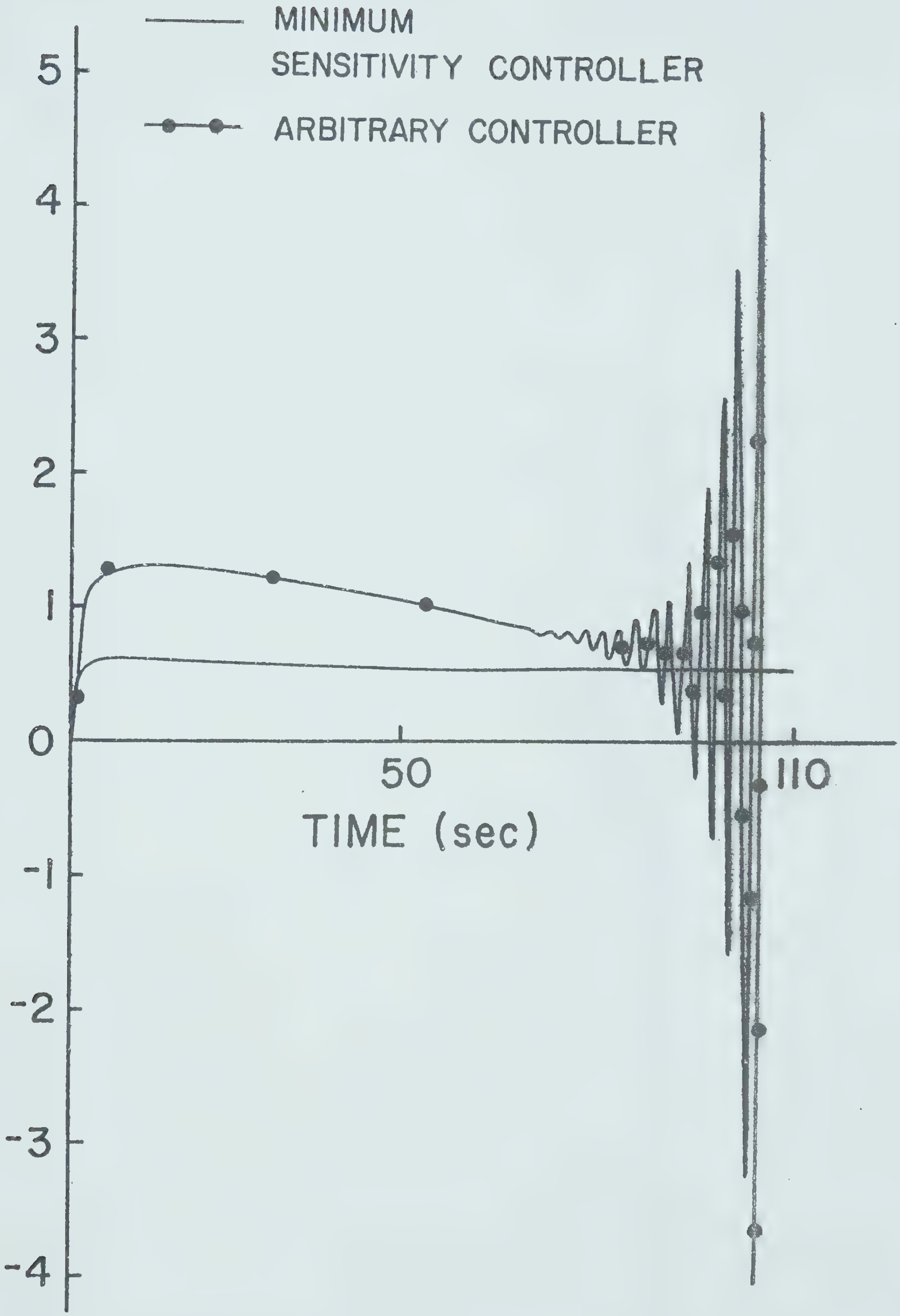

Figure 13. Response of X_1 when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

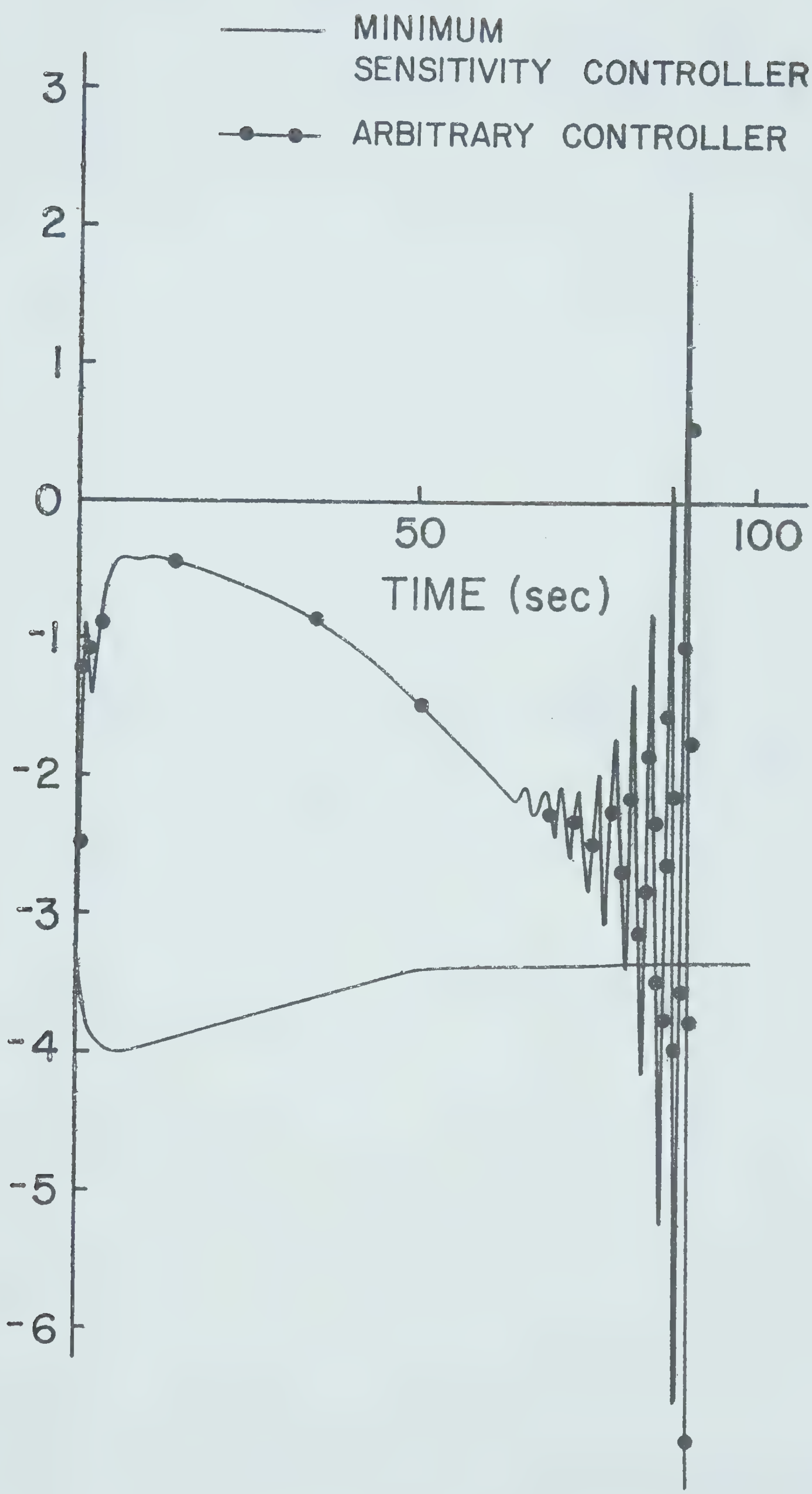

Figure 14. Response of X_2 when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

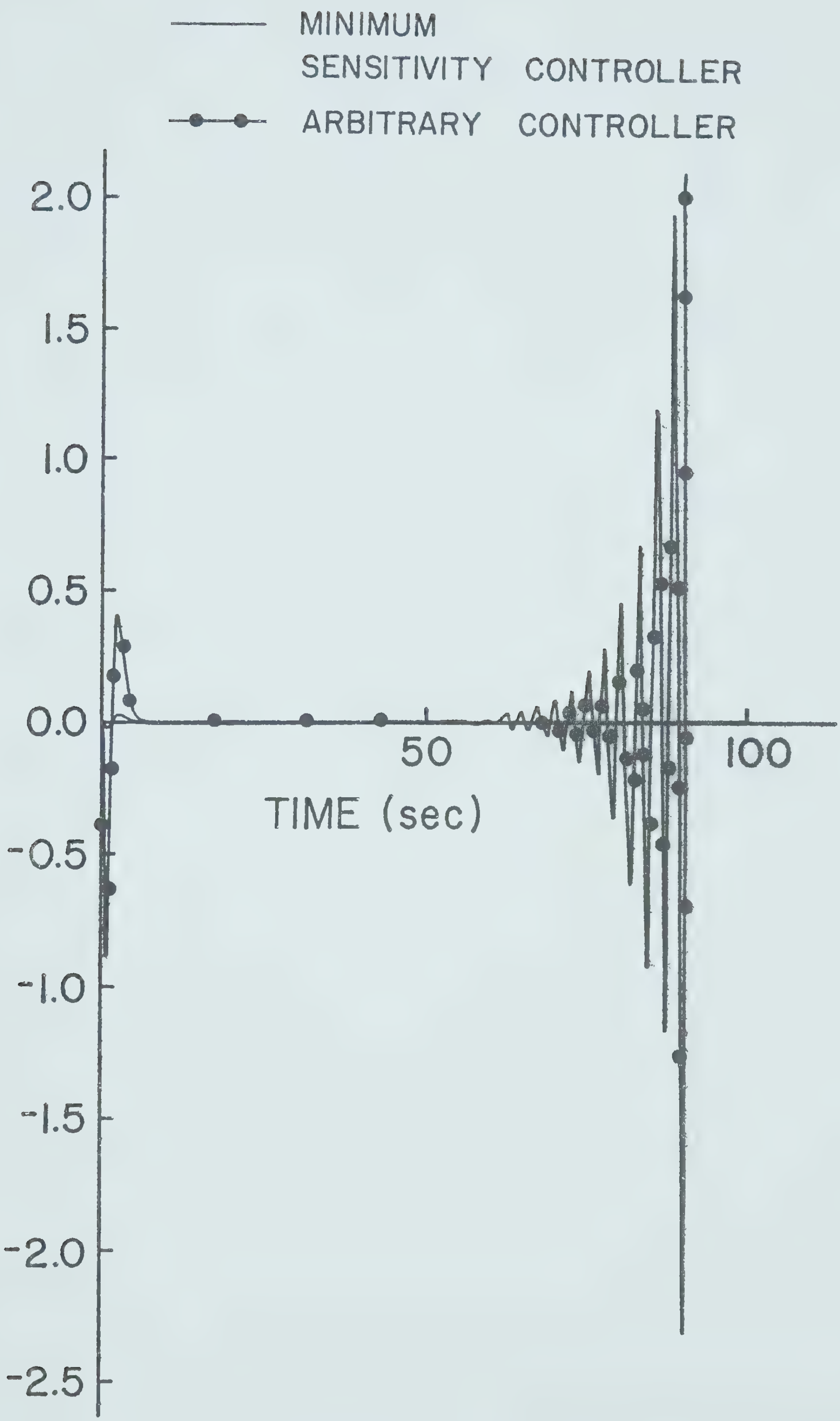

Figure 15. Response of X_3 when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

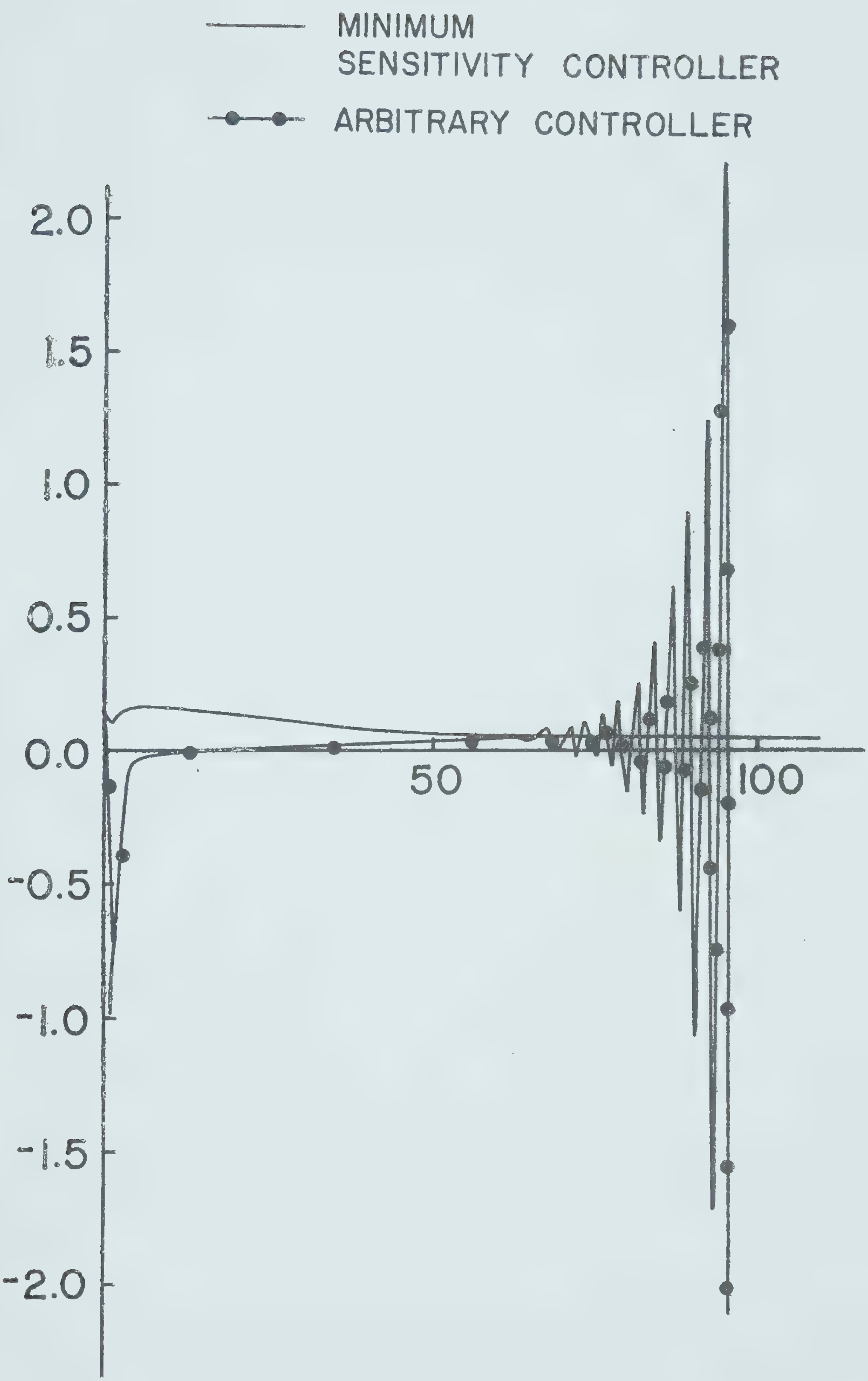

Figure 16. Response of X_4 when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

From equation (3-1) which describes the dynamics of the helicopter in the vertical plane we have designed a unity-rank minimum sensitivity controller of the form shown in equation (2-2). For regulation purposes we will redefine (2-2) to take into account disturbances by defing

$$u = qu' + R + w \text{ ------------------------------------ } (3\text{-}8)$$

where

$u' \equiv$ equation (2-5)

$R \equiv$ pilot input

$w \equiv$ appropriately dimensioned disturbance vector

In all previous calculations and simulations w was taken as identically zero (whether $w = 0$ or $w \neq 0$ has no effect on the design of the controller). It is now desired to see how the controller will regulate the speed of the VTOL aircraft in the presence of disturbances, i.e. $w \neq 0$.

Assume that the pilot is flying the helicopter along in a steady state condition at one hundred and thirty-five knots. Suddenly there is a gust of wind such that $w = [w_1 \quad w_2]^T = [0.4 \quad 0.4]^T$ which lasts for three seconds. Without any corrective pilot action the response of the system variables of the closed-loop system with the minimum sensitivity controller shown in figures 17, 18, 19 and 20 for x_1, x_2, x_3 and x_4 respectively.

For comparison purposes the closed-loop control system with an arbitrary controller having the same pole locations as the minimum sensitivity controller was subjected to the same test. The results are shown

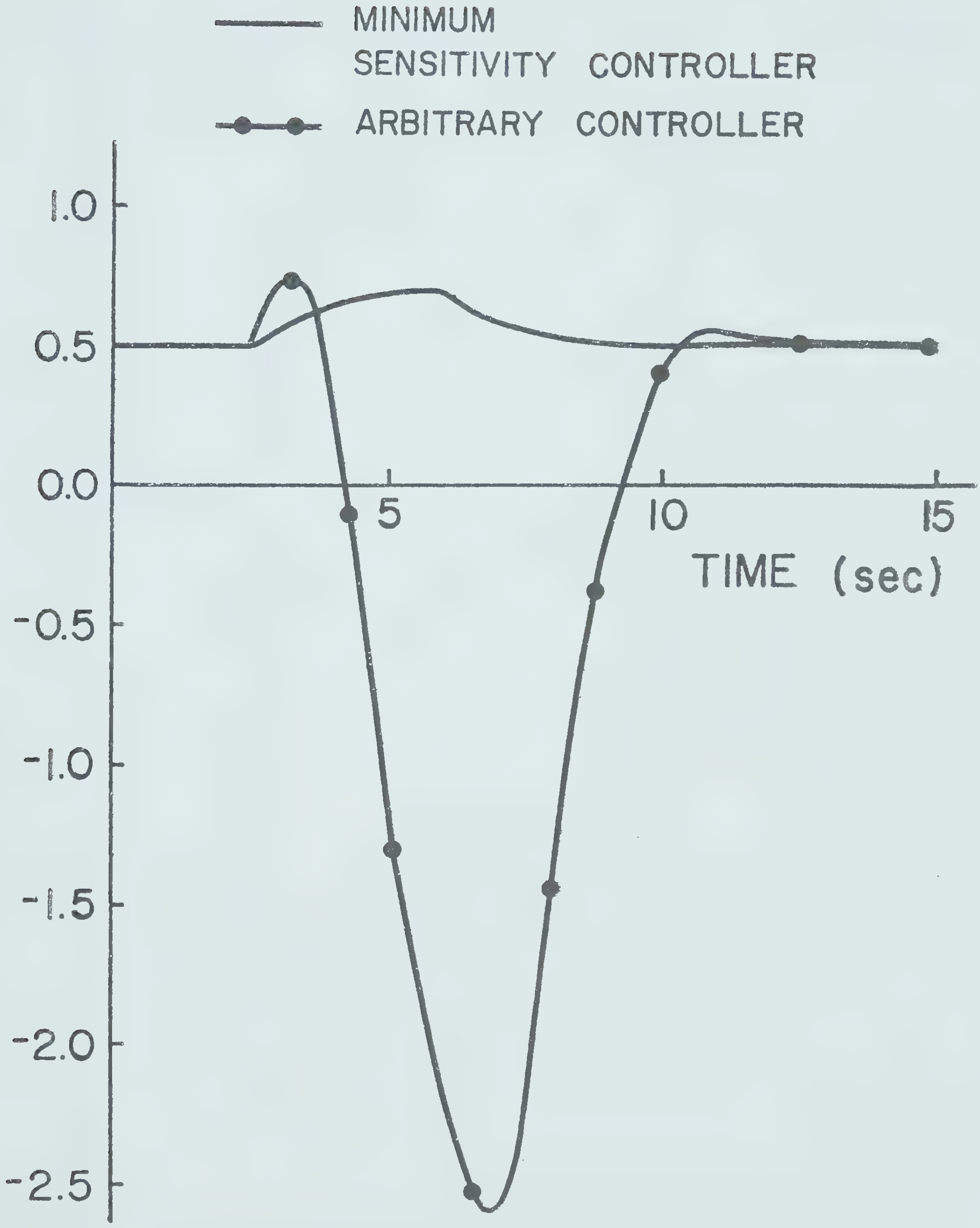

Figure 17. Disturbance Effect on X_1 when Flying in Steady State of One Hundred and Thirty-five Knots

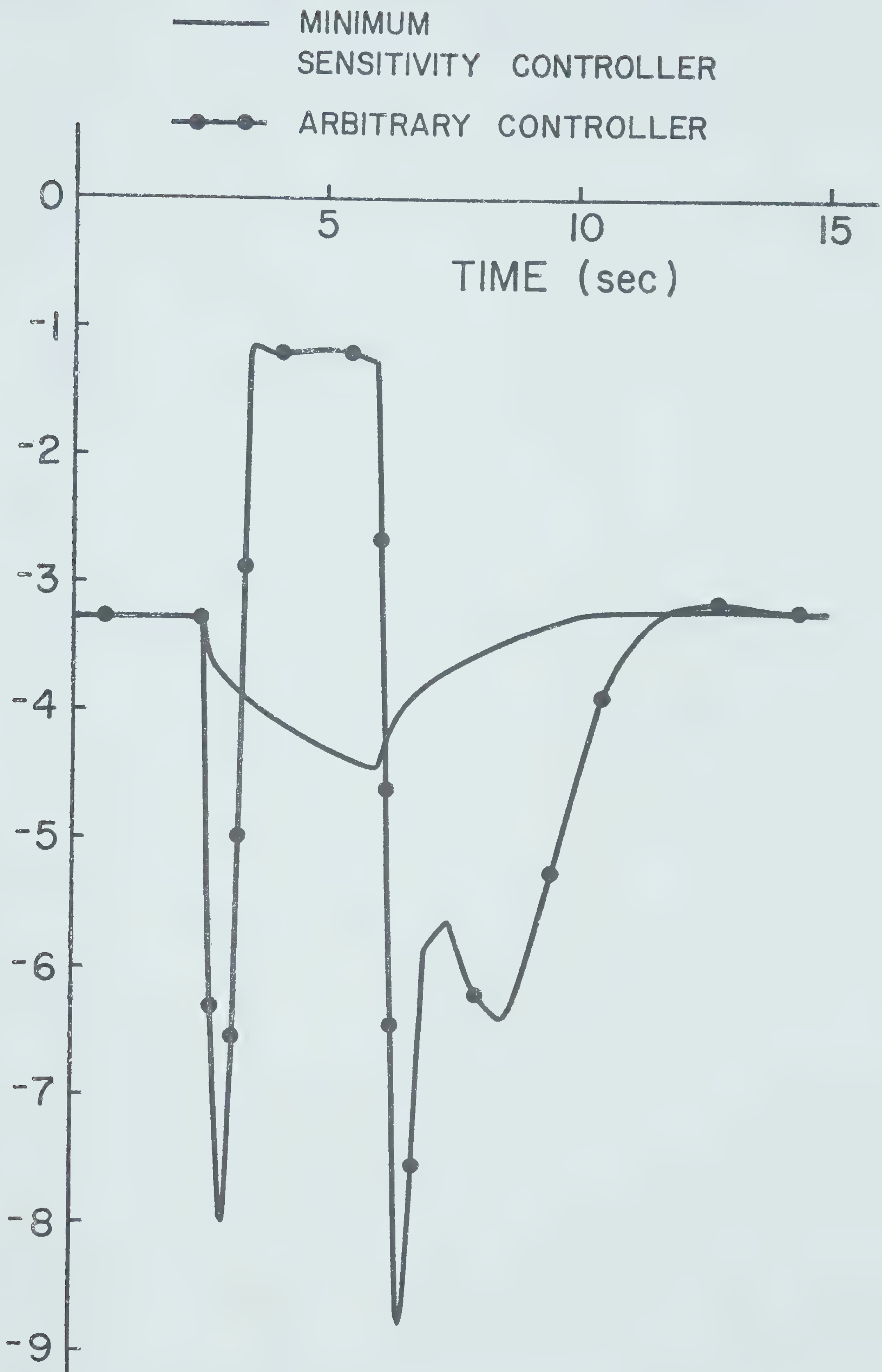

Figure 18. Disturbance Effect on X_2 when Flying in Steady State of One One Hundred and Thirty-five Knots

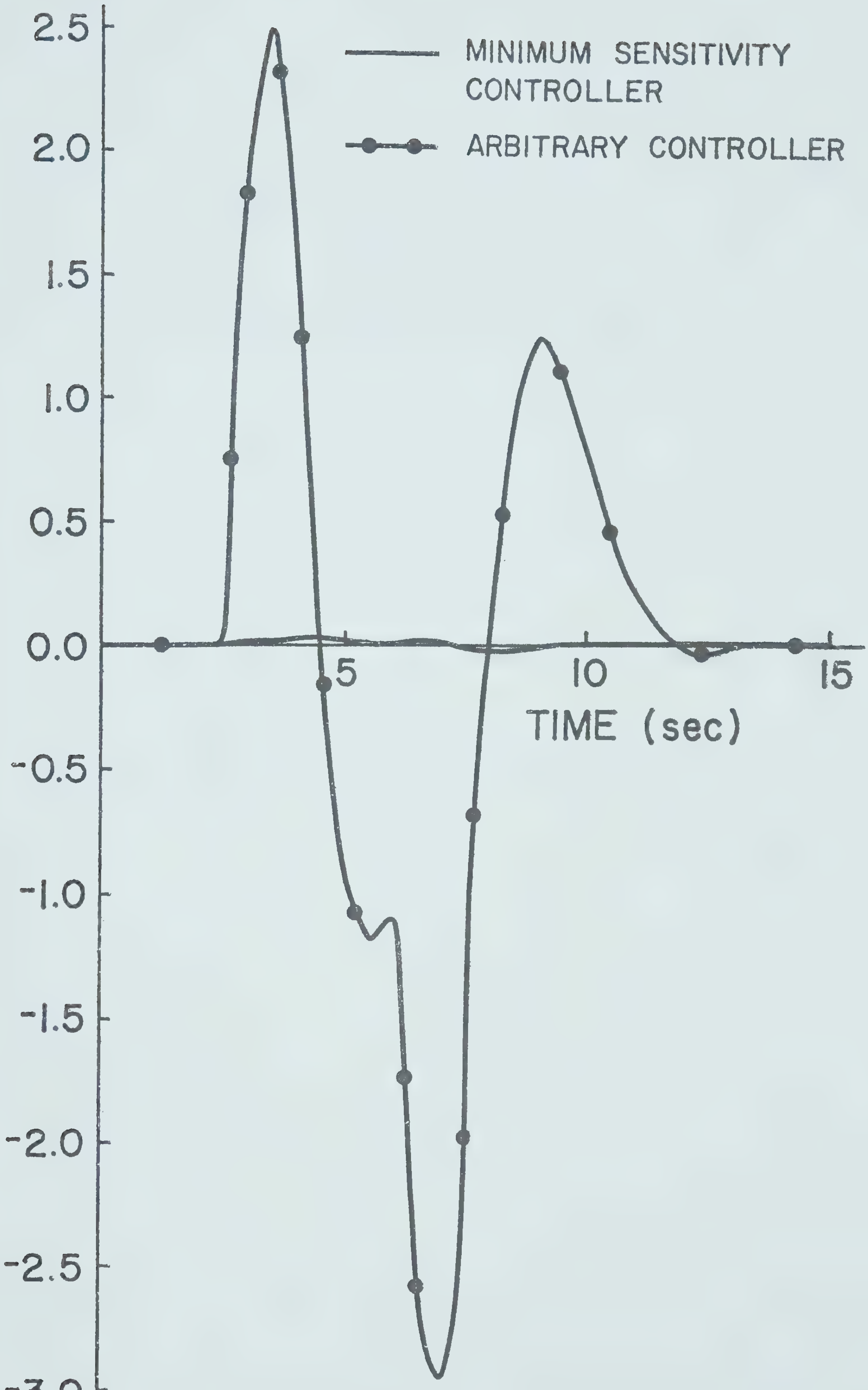

Figure 19. Disturbance Effect on X_3 when Flying in Steady State of One Hundred and Thirty-five Knots

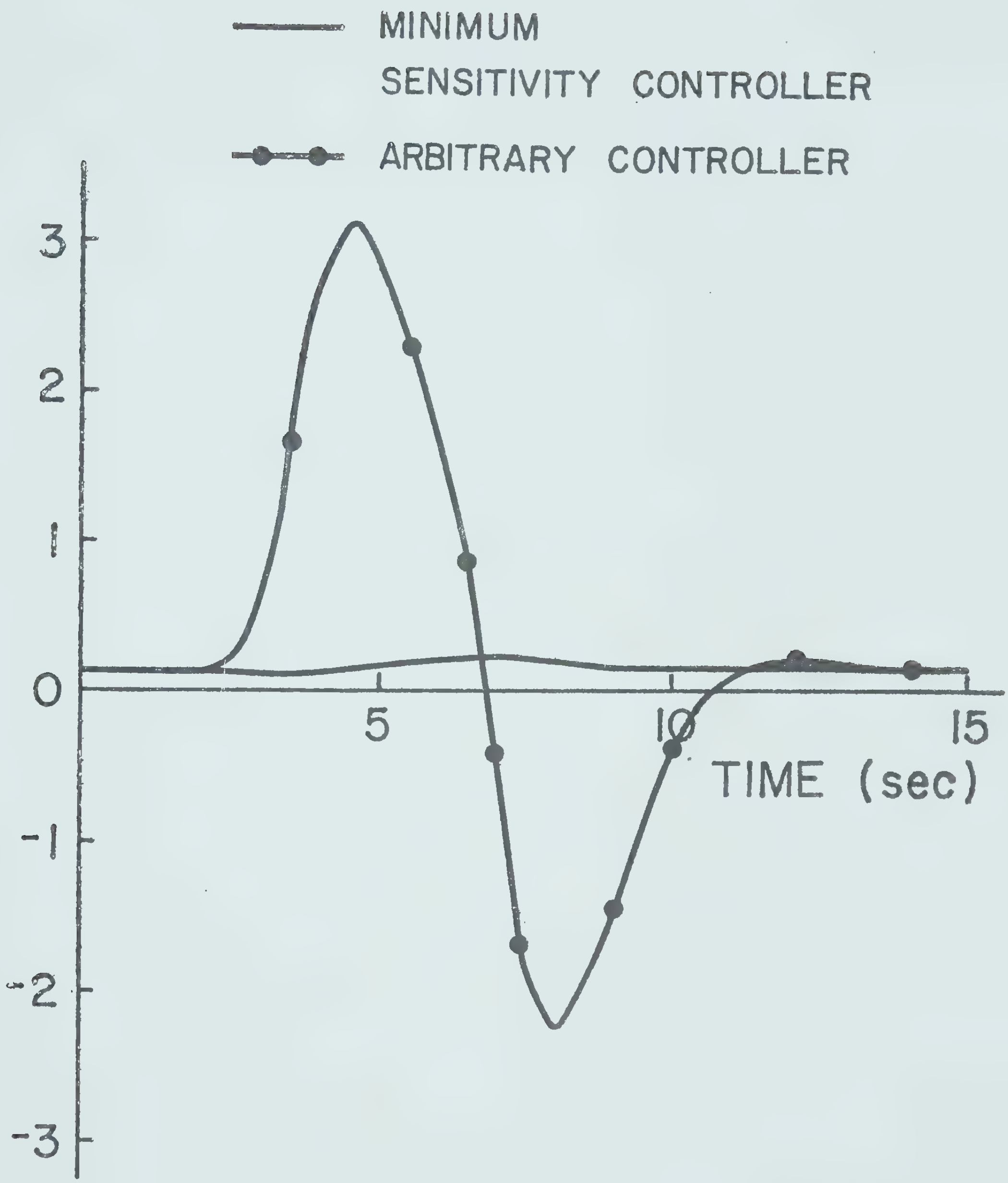

Figure 20. Disturbance Effect on X_4 when Flying in Steady State of One Hundred and Thirty-five Knots

in figures 17, 18, 19 and 20 for x_1, x_2, x_3 and x_4 respectively, to make comparison easier.

From the results shown in figures 17, 18, 19 and 20 it can be seen that the regulating capabilities of the minimum sensitivity controller are far superior to those of the arbitrary controller.

CHAPTER (4)

INCORPORATION OF INTEGRAL CONTROL

4.1 Introduction

In the previous chapter the controller was designed without any consideration given to the steady-state errors produced when the system is subjected to step command inputs. Steady-state errors will always occur when a proportional control system is activated by step command inputs. One way of reducing the steady-state errors is to increase the loop gains considerably, which in general, is undersirable and unacceptable as far as the present work is concerned since the controller is being designed for minimum eigenvalue sensitivity. The steady-state error can also be eliminated by the inclusion of integral control action in the controller. This method is preferable here since integral control can be incorporated in the design procedure.

When the problem was first tackled, it was hoped that the steady-state error of all the four state variables of the helicopter system under study could be made zero. However the present state of the art does not allow the augmentation of the original system by more than the number of inputs to the original system [16]. The system under consideration is of fourth order with two command inputs. Hence the integral control can be applied to not more than two of the four states of the system. In view of this restriction it was decided to apply the integral control to the horizontal (x_1) and the vertical (x_2) velocity of the aircraft system.

4.2 Minimum Sensitivity Controller Design With Zero Steady-state Error

Consider a linear multivariable multi-input dynamic system described by

$$\left.\begin{aligned} \overset{\circ}{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \qquad \text{-----} \quad (4\text{-}1)$$

where

$x \equiv$ n-dimensional state vector;

$u \equiv$ m-dimensional input vector;

$y \equiv$ p-dimensional output vector; and

A, B and C are respectively nxn, nxm and pxn matrices.

It is now desired to augment the original system such that the transient response of the controlled variable(s) is/are satisfactory (response characteristics to satisfy some previous design specifications) and that in the steady-state the output variables become equal to some arbitrary constant reference inputs, i.e , there is no steady-stare error.

Using integral control we introduce new state variables

$$\overset{\circ}{z} = -Cx + Ix_r \qquad \text{-----} \quad (4\text{-}2)$$

where

$z \equiv$ p-dimensional vector;

$x_r \equiv$ p-dimensional state vector to be controlled;and

$I \equiv$ pxp identity matrix.

Combining equations (4-1) and (4-2) the augmented system is

$$\begin{bmatrix} \overset{\circ}{x} \\ \hdashline \overset{\circ}{z} \end{bmatrix} = \left[\begin{array}{c:c} A & 0 \\ \hdashline 0 & 0 \end{array}\right] \begin{bmatrix} x \\ \hdashline z \end{bmatrix} + \left[\begin{array}{c:c} 0 & 0 \\ \hdashline -C & 0 \end{array}\right] \begin{bmatrix} x \\ \hdashline z \end{bmatrix} + \begin{bmatrix} B \\ \hdashline 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \hdashline I \end{bmatrix} x_r \quad \text{--}(4\text{-}3)$$

or

$$\begin{bmatrix} \overset{\circ}{x} \\ \hdashline \overset{\circ}{z} \end{bmatrix} = \left[\begin{array}{c:c} A & 0 \\ \hdashline -C & 0 \end{array}\right] \begin{bmatrix} x \\ \hdashline z \end{bmatrix} + \begin{bmatrix} B \\ \hdashline 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \hdashline I \end{bmatrix} x_r \qquad \text{-----------} \quad (4\text{-}4)$$

where
$$\overset{\circ}{\hat{x}} = A^*x + B^*u + I^*x_r$$

$$A^* = \left[\begin{array}{c:c} A & 0 \\ \hdashline -C & 0 \end{array}\right] \quad ; \quad B^* = \begin{bmatrix} B \\ \hdashline 0 \end{bmatrix} \quad ; \quad I^* = \begin{bmatrix} 0 \\ \hdashline I \end{bmatrix}$$

For the single input case it is a fairly simple matter to choose a feedback matrix that will assign the poles of the closed-loop system to the desired locations if the system given by equation (4-4) is controllable.

For a system of the form (4-4) with more than one input, the system matrix A* will never be cyclic** since it has multiple eigenvalues of zero. Thus it is not possible to choose a unit-rank feedback matrix such that the poles of the closed-loop system will be at the desired locations in the complex frequency plane. Thus the design must proceed in two stages.

First a feedback matrix K_{T1}*** is chosen so that the first stage closed-loop system matrix is cyclic. In order to facilitate this we choose a control law of the form

$$u = u^1 + u^2 + R \qquad \text{------------------------------} \quad (4\text{-}5)$$

where u^1 and u^2 are respectively the first and second stage controls and R is the pilot control input(R is an mx1 input vector). Define the first

---

** The matrix A* will be cyclic if there exists an n-vector w such that the matrix $[w \quad A^*w \quad \ldots \quad A^{*n-1}w]$ is of full rank.

*** Note that K_{T1} can be randonly chosen but KI must be of full rank p. The choice of K_{T1} can be used to satisfy other design specifications not pertaining to minimum sensitivity analysis.

stage control as

$$u^1 = -K_{T1}[x \mid z]^T \quad \text{----------------------------} \quad (4\text{-}6)$$

where

$$K_{T1} = [0 \mid KI] \quad \text{--------------------------------} \quad (4\text{-}7)$$

and when p = m, as in our case, a first stage control matrix of the form shown in (4-7) will generally result in the removal of the non-cyclicity of the system matrix if

$$KI = \begin{bmatrix} ki_{11} & ki_{12} \dots ki_{1p} \\ ki_{21} & ki_{22} \dots ki_{2p} \\ \vdots & \\ ki_{p1} & ki_{p2} \dots ki_{pp} \end{bmatrix} = \begin{bmatrix} 1 & 0 \;.\;.\;.\; 0 \\ 0 & 1 \;.\;.\;.\; 0 \\ \vdots & \vdots \\ 0 & 0 \;.\;.\;.\; 1 \end{bmatrix} \quad \text{-----} \quad (4\text{-}8)$$

and KI is pxp.

Now using equations (4-4), (4-5) and (4-6) results in the first stage closed-loop system equation

$$\begin{bmatrix} \overset{\circ}{x} \\ \hline \overset{\circ}{z} \end{bmatrix} = \left[\begin{array}{c|c} A & -BKI \\ \hline -C & 0 \end{array}\right] \begin{bmatrix} x \\ \hline z \end{bmatrix} + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} u^2 + \begin{bmatrix} B \\ \hline 0 \end{bmatrix} R + \begin{bmatrix} 0 \\ \hline I \end{bmatrix} x_r$$

$$\overset{\circ}{\hat{x}} = \hat{A}^*\hat{x} + B^*u^2 + B^*R + I^*x_r \quad \text{------} \quad (4\text{-}9)$$

where the definition if $\hat{A}^*$ is obvious and B^* and I^* were previously defined. The control scheme is shown in schematic form in figure 21.

Now the first stage closed-loop system matrix, $\hat{A}^*$, of (4-9) can be considered as the open-loop system matrix for the second stage design. The design procedure used in chapter 3 can now be used to assign the poles of the closed-loop augmented system using unity-rank feedback and minimum sensitivity analysis by choosing a second stage control of the form

$$u^2 = -qu' \quad \text{-----------------------------------} \quad (4\text{-}10)$$

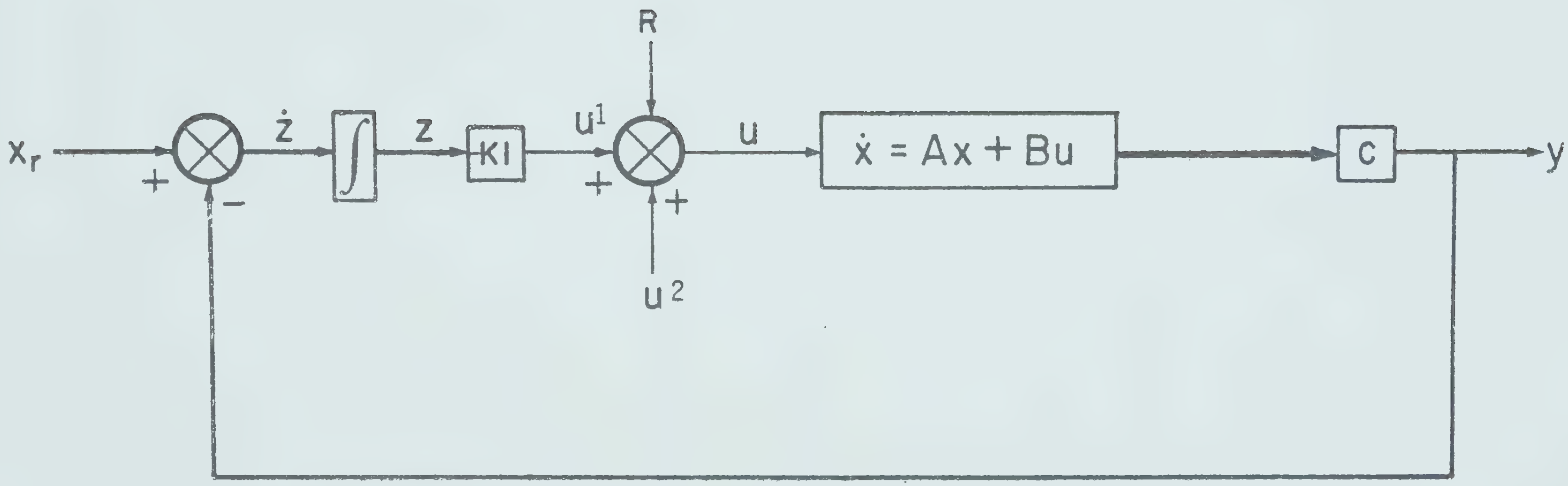

Figure 21. First Stage Control for Integral Controller

where q is the same as in chapter 2 and

$$u' = k_{t2}\left[x \;\vdots\; z\right]^T \quad \text{-------------------------} \quad (4\text{-}11)$$

and k_{t2} is an n+p row vector defined as

$$k_{t2} = \left[kd_{11} \;\dots\; kd_{1n} \;\vdots\; kp_{1n+1} \;\dots\; kp_{1n+p}\right] \quad \text{-----} \quad (4\text{-}12)$$

which we shall denote as

$$k_{t2} = \left[kd \;\vdots\; kp\right] \quad \text{-------------------------------} \quad (4\text{-}13)$$

Then combining (4-10) and (4-11) we have for the second stage control law

$$\left.\begin{aligned} u^2 &= -qk_{t2}\left[x \;\vdots\; z\right]^T \\ \text{or} \qquad u^2 &= -K_{T2}\left[x \;\vdots\; z\right]^T \end{aligned}\right\} \text{---------------------} \quad (4\text{-}14)$$

where K_{T2} is the minimum sensitivity controller defined by

$$K_{T2} = qk_{t2} \quad \text{-----------------------------------} \quad (4\text{-}15)$$

The total control law is then obtained by use of equations (4-5), (4-6) and (4-14) and defined as

$$\left.\begin{aligned} u &= -K_{T1}\left[x \;\vdots\; z\right]^T - K_{T2}\left[x \;\vdots\; z\right]^T + R \\ u &= -\left[K_{T1} + K_{T2}\right]\left[x \;\vdots\; z\right]^T + R \\ \text{or} \qquad u &= -K_T\left[x \;\vdots\; z\right]^T + R \end{aligned}\right\} \text{-----} \quad (4\text{-}16)$$

where

$$K_T = K_{T1} + K_{T2}$$

Recalling the definition of b_q in chapter 2 and noting the definition of K_{T1} and K_{T2} above, we obtain the total closed-loop augmented system equation

$$\begin{bmatrix} \overset{o}{x} \\ \overset{o}{z} \end{bmatrix} = \left[\begin{array}{c|c} A - b_q kd & -BKI - b_q kp \\ \hline -C & 0 \end{array}\right] \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} R + \begin{bmatrix} 0 \\ I \end{bmatrix} x_r$$

$$\overset{o}{\hat{x}} = \hat{A}\hat{x} + B^*R + I^*x_r \qquad \text{----- (4-17)}$$

where the closed-loop system matrix is

$$\hat{A} = \left[\begin{array}{c|c} A - b_q kd & -BKI - b_q kp \\ \hline -C & 0 \end{array}\right] \qquad \text{----------------- (4-18)}$$

The closed-loop poles of the closed-loop system matrix $\hat{A}$ can be placed in the complex frequency plane to any desired locations by the procedure used in chapter 3 if the system of (4-9) is controllable. The system is controllable if and only if:

i) the controllability matrix

$$Q = \left[b_q \,\vdots\, \hat{A}^*b_q \,\vdots\, \dots \,\vdots\, \hat{A}^{*n-1}b_q\right] \qquad \text{--------------- (4-19)}$$

is of full rank n; and

ii) the matrix $\hat{A}^*$ is of rank n + p.

For proof of this see Young and Willems [16].

4.3 Integral Control For VTOL Model

The procedure described in the previous section will now be applied to the VTOL system under study. The horizontal and vertical velocity of the VTOL aircraft are chosen as the output variables to which the integral control is to be applied.

The two new variables used to augment the original system are then described as

$$\overset{o}{x}_5 = -x_1 + x_{r1}$$

$$\overset{o}{x}_6 = -x_2 + x_{r2}$$

where

x_1 = horizontal velocity;

x_2 = vertical velocity;

x_{r1} = reference horizontal velocity; and

x_{r2} = reference vertical velocity.

Stage 1.

We choose a first stage control law as described by equation (4-6) by selecting a first stage control matrix K_{T1} as

$$K_{T1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As in Chapter (3) we again design the controller at the nominal air speed of one hundred and thirty-five knots. With K_{T1} as above the first stage closed-loop system matrix as defined in equation (4-10) is

$$\hat{A}^* = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 & -0.4422 & -0.1761 \\ 0.0482 & -1.01 & 0.0024 & -4.0208 & -3.5446 & 7.5922 \\ 0.1002 & 0.3681 & -0.707 & 1.42 & 5.52 & -4.49 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

We now consider $\hat{A}^*$ as the open-loop system matrix of the augmented helicopter system.

Second Stage.

It is now desired to obtain the second stage controller using minimum sensitivity design to assign the closed-loop poles of the augmented helicopter system to the desired locations in the complex frequency plane of

$$s_1 = -4.0$$

$$s_2 = -3.0$$

$$s_3 = -2.0$$

$$s_4 = -1.5$$

$$s_5 = -1.0 + j1.0$$

$$s_6 = -1.0 - j1.0$$

Since only the parameters a_{32} and a_{34} are varying with changes in the horizontal air speed the minimum eigenvalue sensitivity functional is given by

$$J = \sum_{i=1}^{n+p} (|S^i_{32}|)^2 + (|S^i_{34}|)^2 \quad \text{--------------------- (4-20)}$$

where the sensitivity S^i_{32} and S^i_{34} were defined in Chapter (3).

This functional is a function of q. The Fortram program listed in Appendix I was modified to accommodate the augmented sixth-order system, then used to do the entire design calculations. The minimum value of J occurred for $q = [1 \quad 0.562]^T$. This then resulted in the minimum eigenvalue sensitivity feedback matrix K_{T2} of

$$K_{T2} = \begin{bmatrix} 8.8193 & 0.6028 & -2.1379 & -10.1264 & -9.6836 & 8.3666 \\ 4.9553 & 0.3381 & -1.2012 & -5.6897 & -5.4408 & 4.7009 \end{bmatrix}$$

or the total minimum eigenvalue sensitivity feedback matrix K_T of

$$K_T = \begin{bmatrix} 8.3193 & 0.6028 & -2.1379 & -10.1264 & -8.6836 & 8.3666 \\ 4.9553 & 0.3381 & -1.2012 & -5.6897 & -5.4408 & 5.7009 \end{bmatrix}$$

using the K_T above resulted in the closed-loop system matrix as defined in equation (4-18) of

$$\hat{A} = \begin{bmatrix} -4.8091 & -0.2986 & 1.1757 & 5.0244 & 4.7980 & -4.7036 \\ 6.4084 & -0.5760 & -1.5394 & -11.3237 & -10.5281 & 13.6260 \\ 26.5339 & 2.1718 & -7.1149 & -28.9313 & -23.5041 & 20.5868 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

which has the closed-loop pole locations at the desired positions in the complex frequency plane.

To check the sensitivity of the pole locations to variations of the elements a_{32} and a_{34} of the matrix A, the pole locations of the closed-loop system matrix $\hat{A}$ were computed at the extremes of the flight regime, i.e. at sixty and one hundred and seventy knots, where $a_{32} = 0.06635$, $a_{34} = 0.1198$ and $a_{32} = 0.5047$, $a_{34} = 2.526$ respectively.

At sixty knots: $\Delta a_{32} = -0.30175$ or an 82% change in a_{23} and $\Delta a_{34} = -1.3002$ or a 92% change in a_{34}. The closed-loop poles were found to be

$s_1 = -4.074 + j1.513$

$s_2 = -4.074 - j1.513$

$s_3 = -2.065$

$s_4 = -0.7652$

$s_5 = -0.7606 + j1.589$

$s_6 = -0.7606 - j1.589$

which indicate pole location changes of

$\Delta s_1 = -0.074 + j1.513$ or a 38% change in s_1 @ $\underline{/-20}.4^{o}$

$\Delta s_2 = -1.074 - j1.513$ or a 62% change in s_2 @ $\underline{/20}.4^{o}$

$\Delta s_3 = -0.065$ or a 3.2% change in s_3

$\Delta s_4 = 0.7348$ or a 49% change in s_4

$\Delta s_5 = 0.3494 + j0.589$ or a 48% change in s_5 @ $\underline{/-19}.4^{o}$

$\Delta s_6 = 0.3494 - jo.589$ or a 48% change in s_5 @ $\underline{/19}.4^{o}$

At one hundred and seventy knots: $\Delta a_{32} = 0.1366$ indicating a change in a_{32} of 37% and $\Delta a_{34} = 1.106$ indicating a 78% change in a_{34}. The closed-loop poles were found to be

$s_1 = -5.03$

$s_2 = -2.159 + j1.683$

$s_3 = -2.159 - j1.683$

$s_4 = -2.075$

$s_5 = -0.5367 + j0.6379$

$s_6 = -0.5367 - j0.6379$

which indicate pole location changes of

$\Delta s_1 = -1.03$ or a 25.7% change in s_1

$\Delta s_2 = -0.941 + j1.683$ or a 64% change in s_2 @ $\underline{/7^{o}}$

$$\Delta s_3 = -0.159 - j1.683 \quad \text{or an 84\% change in } s_3 \ @\underline{/-7^0}$$

$$\Delta s_4 = -0.575 \quad \text{or a 38\% change in } s_4$$

$$\Delta s_5 = 0.4633 - j0.3721 \text{ or a 42\% change in } s_5 \ @ \ \underline{/-4.9^0}$$

$$\Delta s_6 = 0.4633 + j0.3721 \text{ or a 42\% change in } s_6 \ @ \ \underline{/4.9^0}$$

For comparison an arbitrary controller was designed to assign the poles of the closed-loop system to the same locations as before. Choice of $q = [1 \quad 3]^T$ resulted in the arbitrary feedback controller of

$$K_{T2} = \begin{bmatrix} 13.3175 & -1.2647 & -3.3333 & -7.1020 & -15.3930 & 3.4701 \\ 39.9526 & -3.7940 & -10.0000 & -21.3059 & -46.1789 & 10.4103 \end{bmatrix}$$

when K_{T1} was the same as in the minimum sensitivity design. The complete feedback controller is

$$K_T = \begin{bmatrix} 13.3175 & -1.2647 & -3.3333 & -7.1020 & -14.3930 & 3.4701 \\ 39.9526 & -3.7940 & -10.0000 & -21.3059 & -46.1789 & 11.4103 \end{bmatrix}$$

The arbitrary controller above resulted in the closed-loop system matrix of

$$\hat{A} = \begin{bmatrix} -12.9613 & 1.2544 & 3.2538 & 6.4370 & 14.4967 & -3.5438 \\ 256.1712 & -25.3318 & -64.1044 & -140.6057 & -299.5822 & 74.3289 \\ -105.7743 & 10.4231 & 25.7931 & 57.8806 & 127.8941 & -32.0772 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ -1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & -1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

which has the desired pole locations.

Again checking the pole locations of the system when using the arbitrary controller and subjecting A to the same variations as in the

minimum sensitivity case, it was found that at:

Sixty knots

$s_1 = -9.326$

$s_2 = -1.85$

$s_3 = -0.4069 + j0.5479$

$s_4 = -0.4069 - j0.5479$

$s_5 = -0.2555 + j3.386$

$s_6 = -0.2555 - j3.386$

resulting in a

$\Delta s_1 = -5.326$ or a 133% change in s_1

$\Delta s_2 = 2.15$ or a 72% change in s_2

$\Delta s_3 = 1.5931 + j0.5479$ or an 84% change in s_3 @ $\angle -8.4^{\circ}$

$\Delta s_4 = 1.0931 - j0.5479$ or an 82% change in s_4 @ $\angle 8.4^{\circ}$

$\Delta s_5 = 0.7445 + j2.386$ or a 177% change in s_5 @ $\angle -40.8^{\circ}$

$\Delta s_6 = 0.7445 - j2.386$ or a 177% change in s_6 @ $\angle 40.8^{\circ}$

One hundred and seventy knots

$s_1 = -4.955 + j4.145$

$s_2 = -4.955 - j4.145$

$s_3 = -1.936$

$s_4 = -0.4509$

$s_5 = -0.086 + j1.177$

$s_6 = -0.086 - j1.177$

resulting in a

$\Delta s_1 = -0.955 + j4.145$ or a 106% change in s_1 @ $\angle -40^{\circ}$

$\Delta s_2 = -1.955 - j4.145$ or a 153% change in s_2 @ $\angle 40^{\circ}$

$\Delta s_3 = 0.064$ or a 3.2% change in s_3

$\Delta s_4 = 1.019$ or a 70% change in s_4

$\Delta s_5 = 0.914 + j0.177$ or a 66% change in s_5 @ $\underline{/-40}.8^0$

$\Delta s_6 = 0.914 - j0.177$ or a 66% change in s_6 @ $\underline{/40}.8^0$

A comparison of the above results reveal that the variations in the poles of the closed-loop system, due to changes in the system plant parameters, are appreciably less when the minimum sensitivity design approach is used to obtain the feedback controller.

4.4 Tracking Capabilities of the Zero Steady-state Error Controller

To illustrate the tracking capabilities of the two controllers designed in section 4.3 the system was simulated on the digital computer using the program listed in appendix II. The simulation here is slightly different than that of chapter 3.

Now a prior knowledge of the command inputs must be known [16,37] ,i.e. x_r must be known. Then in the steady-state the controlled outputs (the outputs to which the integral control action is applied, in this case x_1 and x_2) should approach the values of x_{r1} and x_{r2} respectively, i.e., the steady-state error should be zero.

The responses of both systems were obtained when accelerating from sixty to one hundred and thirty-five knots and again when accelerating from one hundred and thirty-five to one hundred and seventy knots. The following steps were involved in the respective simulations:

i) to begin the simulation at sixty knots

a) $R = [0.4 \quad 0.4]^T$ was arbitrarily chosen;

b) for x_{r1} and x_{r2} equal to 1.0 and 0.0 respectively

the steady-state values are obtained for the other variables as, $x_1 = 1.0$, $x_2 = 0.0$, $x_3 = 0.0$ and $x_4 = -0.018$.

ii) to accelerate from sixty to one hundred and thirty-five knots

a) $R = [1.0 \quad 1.0]^T$ was chosen;

b) initial conditions were set as obtained in (i); and

c) x_{r1} and x_{r2} equaled 3.0 and 0.0 respectively.

iii) to accelerate from one hundred and thirty-five to one hundred and seventy knots

a) $R = [1.25 \quad 1.25]^T$ was chosen;

b) initial conditions were set as obtained in (ii); and

c) x_{r1} and x_{r2} equaled 5.0 and 0.0 respectively.

A schematic diagram of the control system used for the above simulations is shown in figure 22.

The responses of the horizontal (x_1) and vertical (x_2) velocities are shown in figure 23 for x_1 and figure 25 for x_2 when accelerating from sixty to one hundred and thirty-five knots and in figure 24 for x_1 and figure 26 for x_2 when accelerating from one hundred and thirty-five to one hundred and seventy knots for both the minimum sensitivity and the arbitrary controllers.

4.5 <u>Regulating Capabilities of the Zero Steady-state Error Controllers</u>

In this section we look at the ability of the controllers to maintain a constant output, i.e. zero steady-state error of the chosen outputs, when the system is subjected to disturbances.

The original equation (4-1) is rewritten here to include the disturbance vector w as

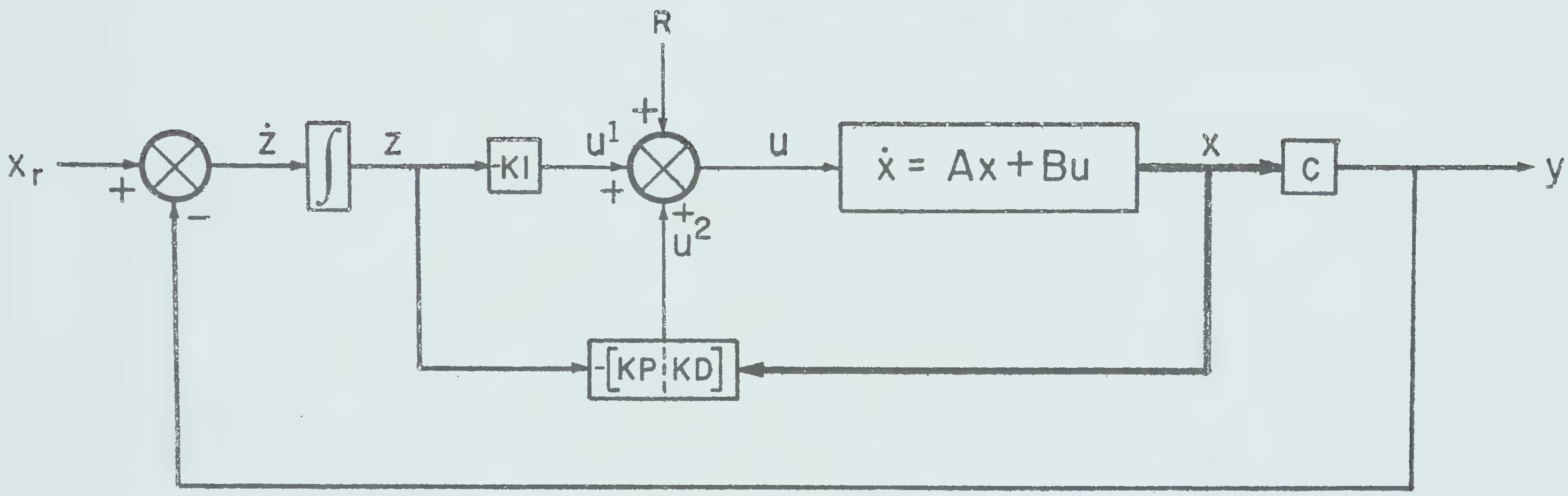

Figure 22. First and Second Stage (Full) Control Scheme of Integral Controller for VTOL System

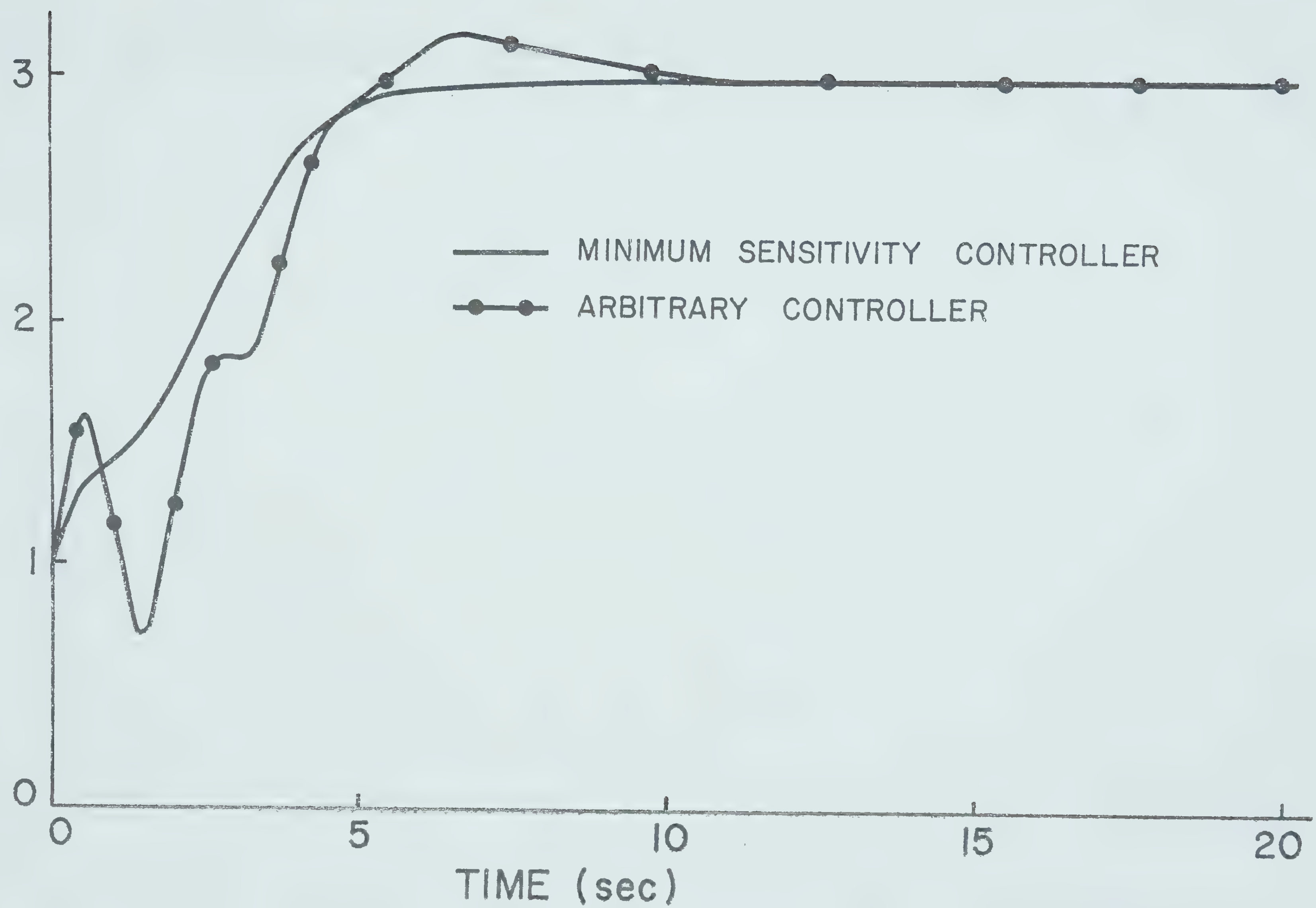

Figure 23. Response of X_1 Using Integral Control when Accelerating from Sixty to One Hundred and Thirty-five Knots

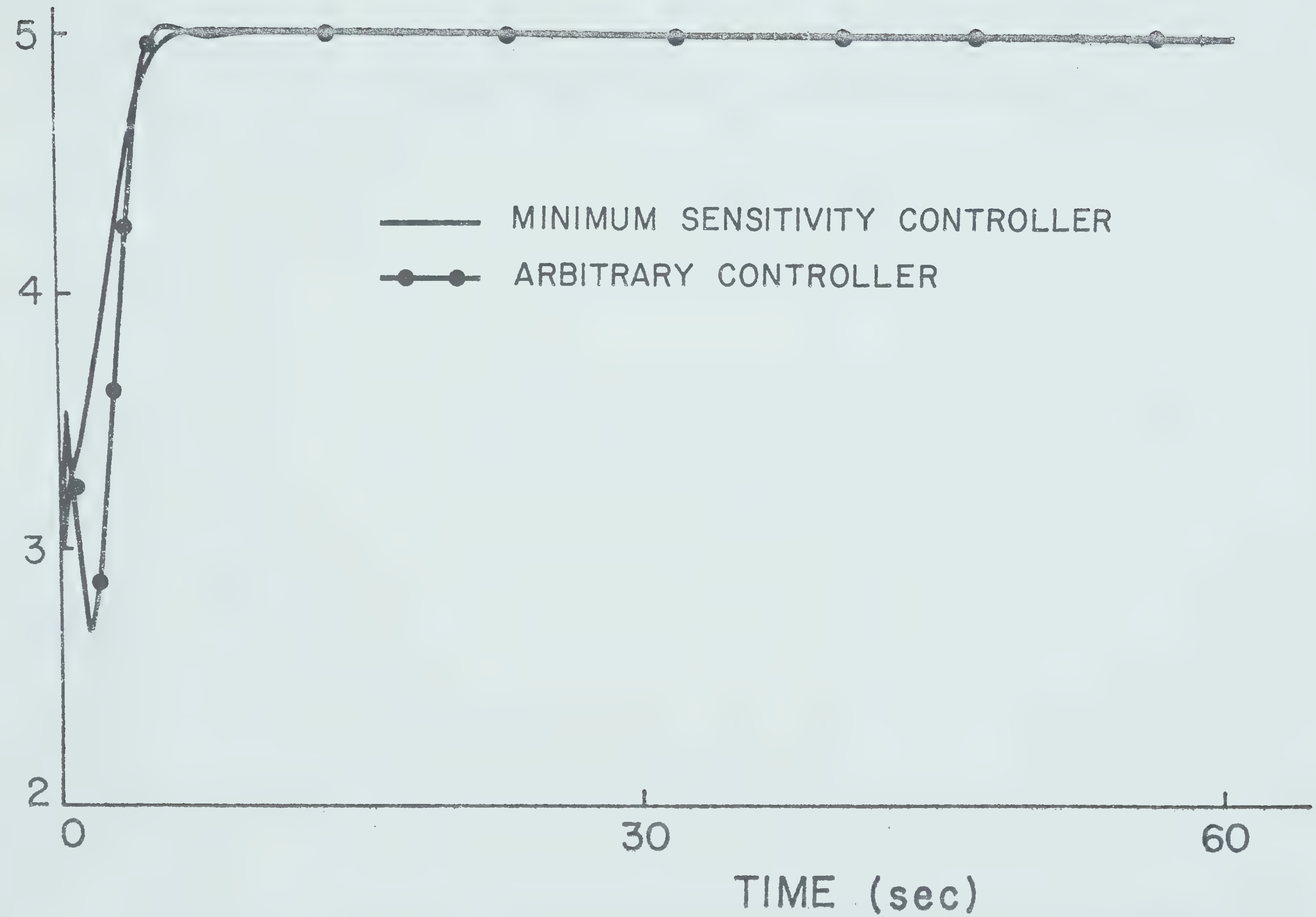

Figure 24. Response of X_1 Using Integral Control when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

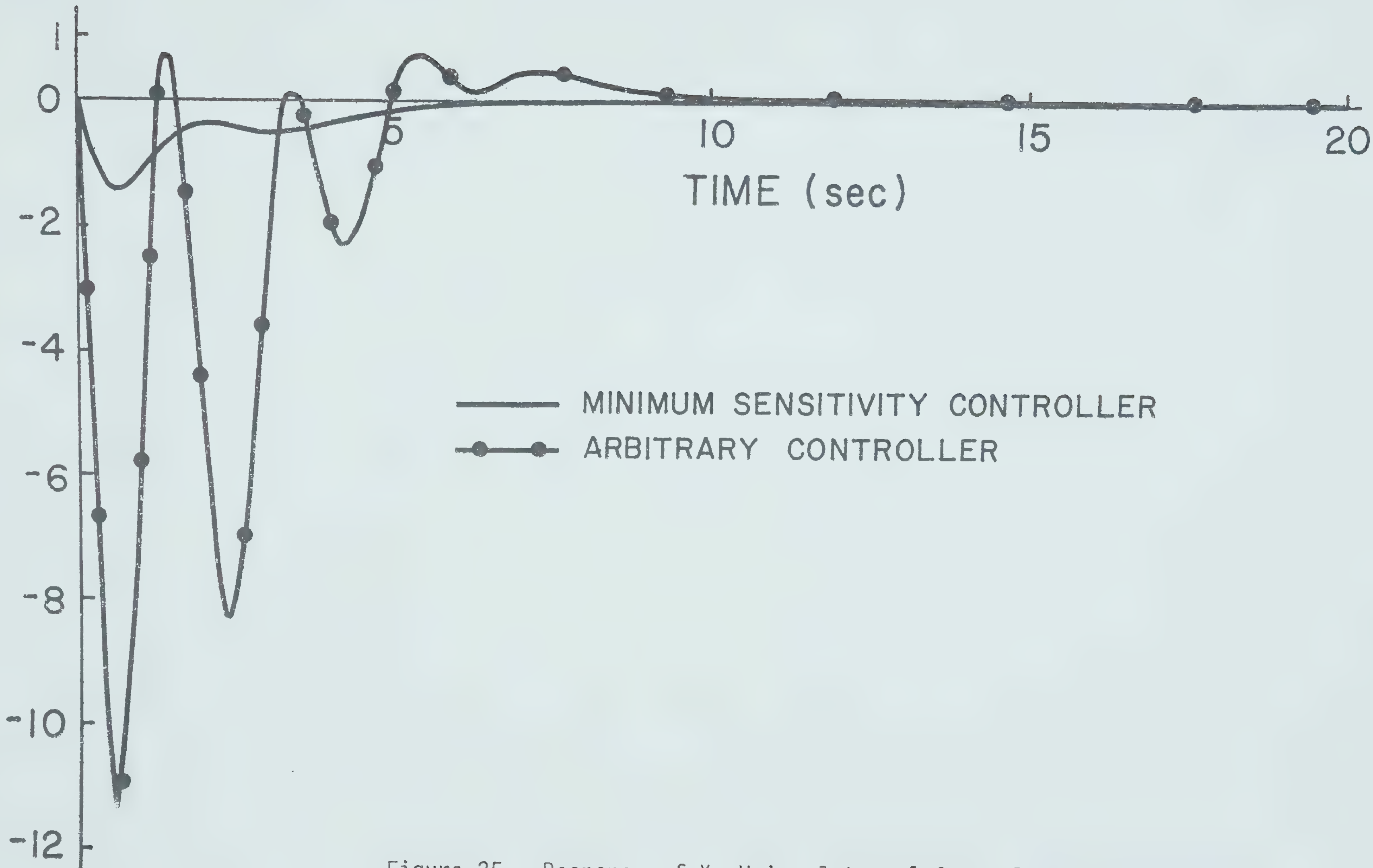

Figure 25. Response of X_2 Using Integral Control when Accelerating from Sixty to One Hundred and Thirty-five Knots

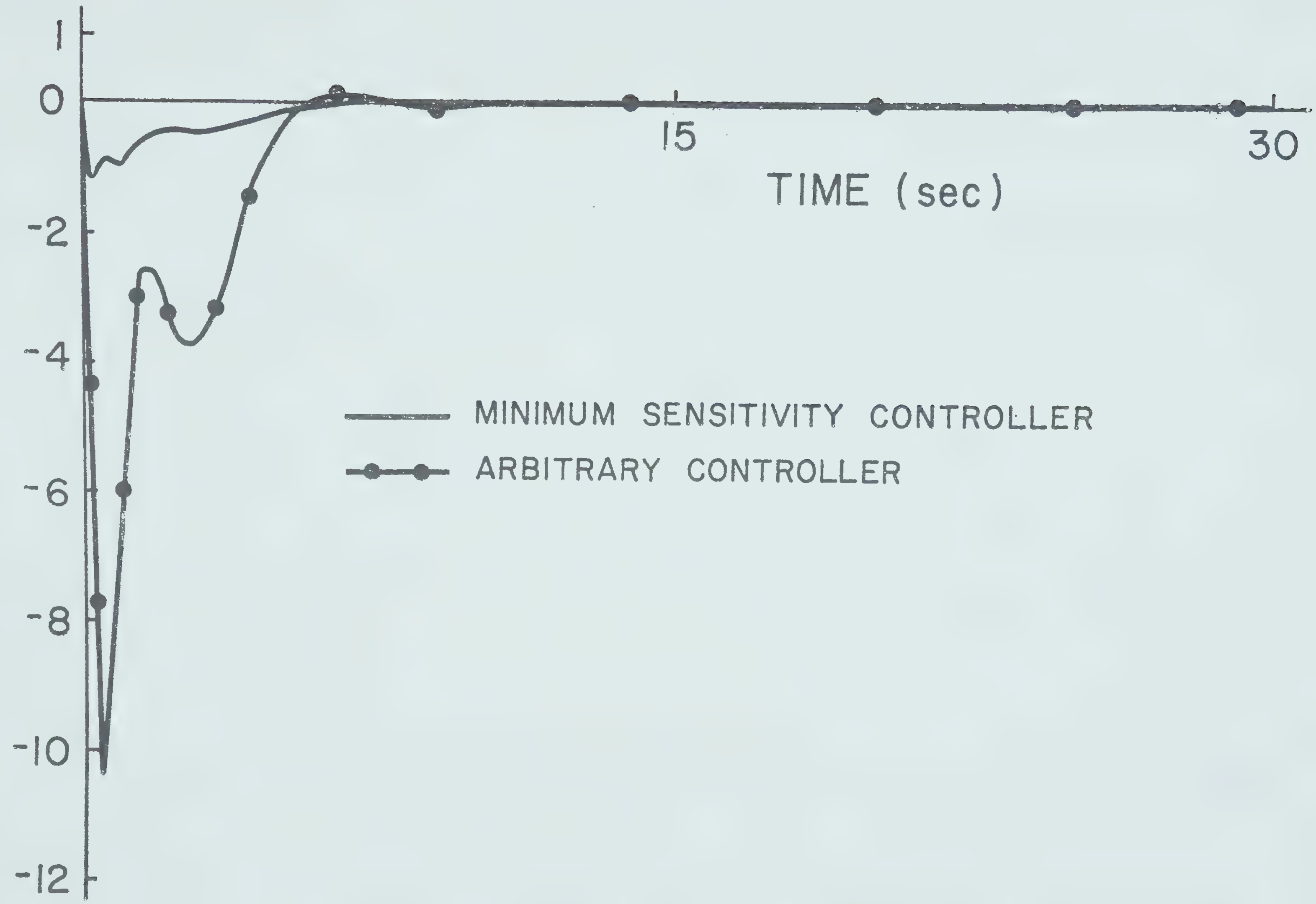

Figure 26. Response of X_2 Using Integral Control when Accelerating from One Hundred and Thirty-five to One Hundred and Seventy Knots

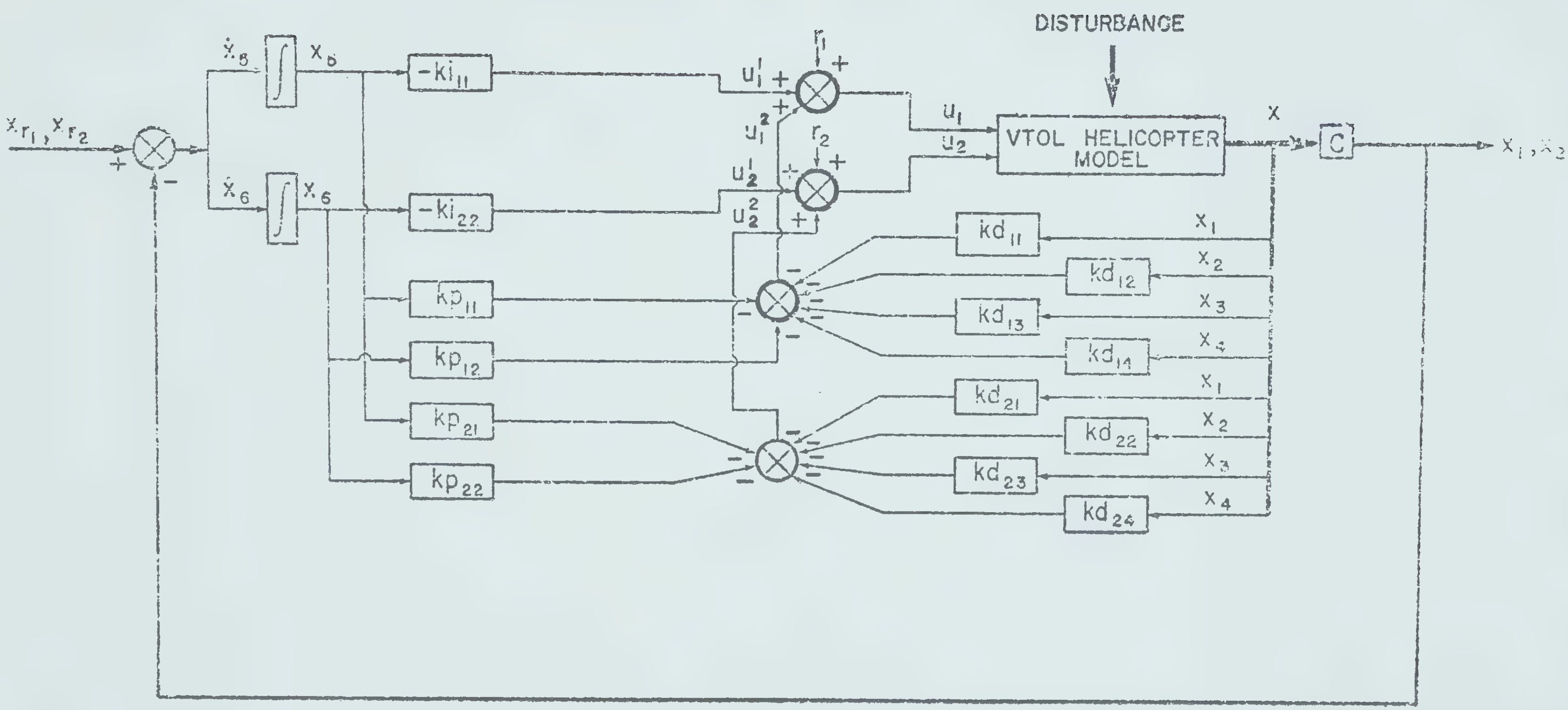

Figure 27. Integral Control Scheme of VTOL System with Variable Disturbance

$$\overset{\circ}{x} = Ax + Bu + Bw \quad \text{------------------------------} \quad (4\text{-}21)$$

where w is previously defined.

Let $\bar{x}$, $\bar{u}$, $\bar{w}$ be the nominal values of the respective variables and $\bar{A}$ be the nominal value of the system matrix A. In terms of the deviations of the variables from their nominal values let

$$\left.\begin{aligned} \overset{\circ}{x} &\to \overset{\circ}{\bar{x}} + \delta\overset{\circ}{x} \\ x &\to \bar{x} + \delta x \\ u &\to \bar{u} + \delta u \\ w &\to \bar{w} + \delta w \\ A &\to \bar{A} + \delta A \end{aligned}\right\} \quad \text{------------------------------} \quad (4\text{-}22)$$

Then equation (4-21) in terms of (4-22) becomes

$$\overline{\overset{\circ}{\mathbf{x}}} + \delta\overset{\circ}{\mathbf{x}} = (\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x}) + \mathbf{B}(\mathbf{u} + \delta u) + \mathbf{B}(\mathbf{w} + \delta\mathbf{w}) \quad \text{----} \quad (4\text{-}23)$$

or the perturbed equation in terms of the deviations of the variables is

$$\delta\overset{\circ}{x} = (A + \delta A)\delta x + B\delta u + B\delta w + (\delta A)\bar{x} \quad \text{--------------} \quad (4\text{-}24)$$

where

$$\delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \delta a_{32} & & \delta a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad ; \quad \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix}$$

$$\delta x = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \end{bmatrix} \quad ; \quad \delta u = \begin{bmatrix} \delta u_1 \\ \delta u_2 \end{bmatrix} \; ; \; \delta w = \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

We introduce new state variables $\delta\overset{\circ}{x}_5$ and $\delta\overset{\circ}{x}_6$ described by

$$\begin{bmatrix} \delta \overset{\circ}{x}_5 \\ \delta \overset{\circ}{x}_6 \end{bmatrix} = \begin{bmatrix} \delta x_{r1} - \delta x_1 \\ \delta x_{r2} - \delta x_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \delta \overset{\circ}{x}_5 \\ \delta \overset{\circ}{x}_6 \end{bmatrix} = \begin{bmatrix} -\delta x_1 \\ -\delta x_2 \end{bmatrix} \qquad \text{-------------------------------} \quad (4\text{-}25)$$

(since x_{r1} and x_{r2} are the reference values and therefore $\delta x_{r1} \equiv \delta x_{r2} \equiv 0.$) in order to maintain the deviation δx_1 and δx_2 of the horizontal and vertical velocity at zero in steady state. The augmented system then becomes

$$\delta \overset{\circ}{\tilde{x}} = \tilde{A}\delta\tilde{x} + \tilde{B}\delta u + \tilde{B}\delta w + (\widetilde{\delta A})\overline{\tilde{x}} \qquad \text{-----------------} \quad (4\text{-}26)$$

where

$$\delta\tilde{x} = \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \\ \delta x_4 \\ \delta x_5 \\ \delta x_6 \end{bmatrix} \quad ; \quad \tilde{A} = \left[\begin{array}{c|c} A + \delta A & 0 \\ \hline -C & 0 \end{array}\right]$$

$$\tilde{B} = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] \qquad \widetilde{\delta A} = \left[\begin{array}{c|c} \delta A & 0 \\ \hline 0 & 0 \end{array}\right] \qquad \overline{\tilde{x}} = \left[\begin{array}{c} x \\ \hline 0 \end{array}\right]$$

Equation (4-23) represents the system simulated on the computer at the nominal airspeed of one hundred and thirty-five knots with the perturbed control law δu of

$$\delta u = -K_T \delta\tilde{x} \qquad \text{-------------------------------------} \quad (4\text{-}27)$$

where K_T is defined in section 4.2 and has the form

$$K_T = \begin{bmatrix} kd_{11} & kd_{12} & kd_{13} & kd_{14} & (ki_{11} + kp_{11}) & (ki_{12} + kp_{12}) \\ kd_{21} & kd_{22} & kd_{23} & kd_{24} & (ki_{21} + kp_{21}) & (ki_{22} + kp_{22}) \end{bmatrix}$$

Since $u = \bar{u} + \delta u$ we have for the control law

$$\begin{aligned} u_1 &= \bar{u}_1 + \delta u_1 \\ u_2 &= \bar{u}_2 + \delta u_2 \end{aligned} \qquad \text{------------------------------} \quad (4\text{-}28)$$

where

$$\delta u_1 = -kd_{11}\delta x_1 - kd_{12}\delta x_2 - kd_{13}\delta x_3 - kd_{14}\delta x_4 - (ki_{11} + kp_{11})\delta x_5 + (ki_{12} + kp_{12})\delta x_6$$

$$\delta u_2 = -kd_{21}\delta x_1 - kd_{22}\delta x_2 - kd_{23}\delta x_3 - kd_{24}\delta x_4 - (ki_{21} + kp_{21})\delta x_5 + (ki_{22} + kp_{22})\delta x_6$$

The control scheme is shown in figure 27 where $\bar{u}$ is replaced by the arbitrarily chosen pilot input R where R is taken as

$$R = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}$$

For the simulation the nominal values of $\bar{A}$ and B were taken from Narendra and Tripathi [1] , and are listed in section 3.3 . For a desired horizontal velocity of one hundred and thirty-five knots and zero vertical velocity the following nominal values were selected

i) pilot inputs

$$r_1 = 1.0$$
$$r_2 = 1.0$$

ii) reference values

$$x_{r1} = 3.0$$
$$x_{r2} = 0.0$$

For the desired horizontal velocity of $x_1 = 3.0$ and vertical velocity of $x_2 = 0.0$ the following steady-state values were obtained for the other variables when:

i) using the minimum sensitivity controller

a) $x_1 = 3.0$

$x_2 = 0.0$

$x_3 = 0.0$

$x_4 = -0.067$

b) u_1(collective) = 0.132

u_2(longitudinal cyclic) = 0.116

ii) using the arbitrary controller

a) $x_1 = 3.0$

$x_2 = 0.0$

$x_3 = 0.0$

$x_4 = -0.067$

b) u_1(collective) = 0.132

u_2(longitudinal cyclic) = 0.116

Both systems were then simulated and subjected to the same disturbance vector $\delta w = [\delta w_1 \ \ \delta w_2]^T$ as shown in figure 28. The responses of x_1 and x_2 for both systems are shown in figure 29 and figure 30 respectively.

Upon comparison of the responses, x_1 and x_2, it is seen that the variations produced when using the minimum sensitivity controller are somewhat less than those produced when the arbitrary controller is used.

It is also noted that when checking the responsed of x_1 and x_2 in all the figures of this chapter, that because of the incorporation of integral control for the variables x_1 and x_2, there is no steady-state error in any case (i.e., in steady-state the outputs x_1 and x_2 equal the reference inputs x_{r1} and x_{r2} respectively).

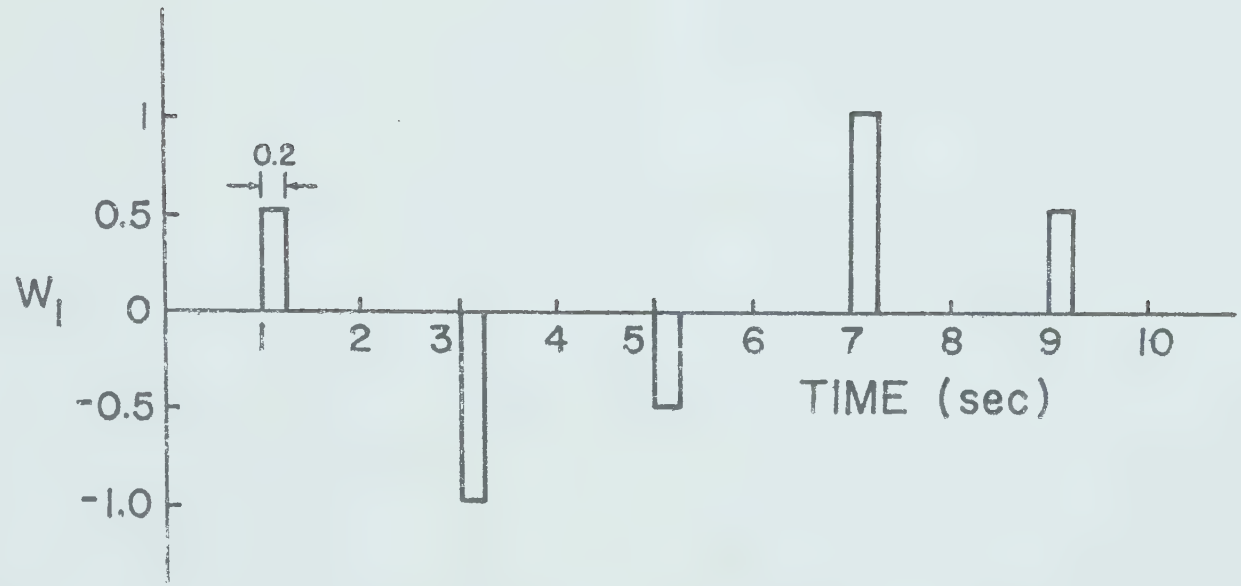

Figure 28(a). Disturbance (w_1) as a Function of Time: Applied to Integral Controller

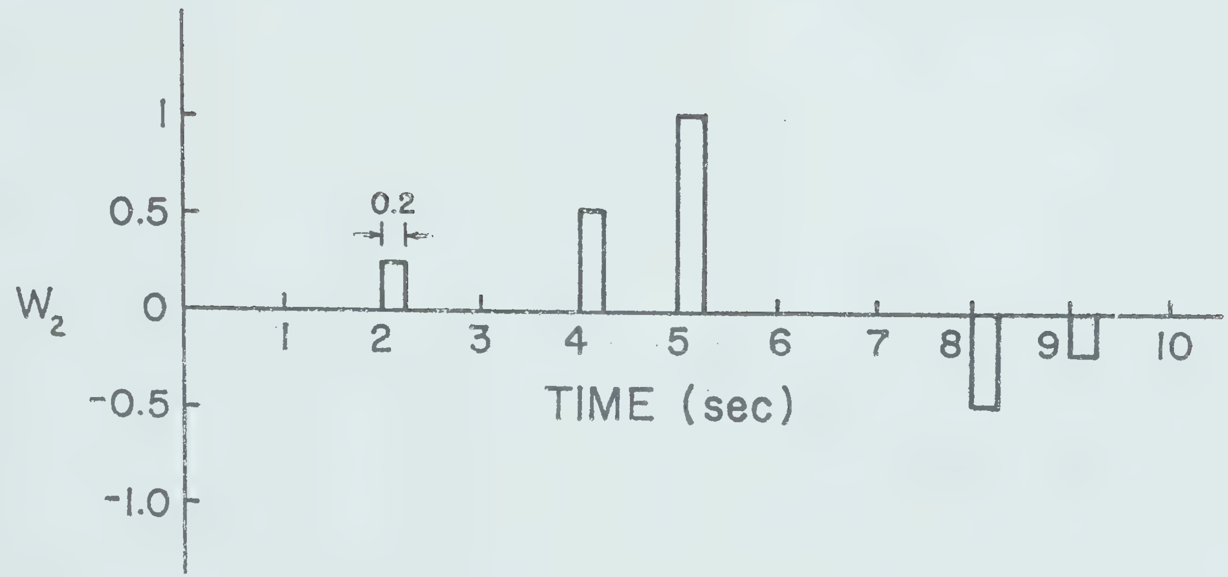

Figure 28(b). Disturbance (w_2) as a Function of Time: Applied to Integral Controller

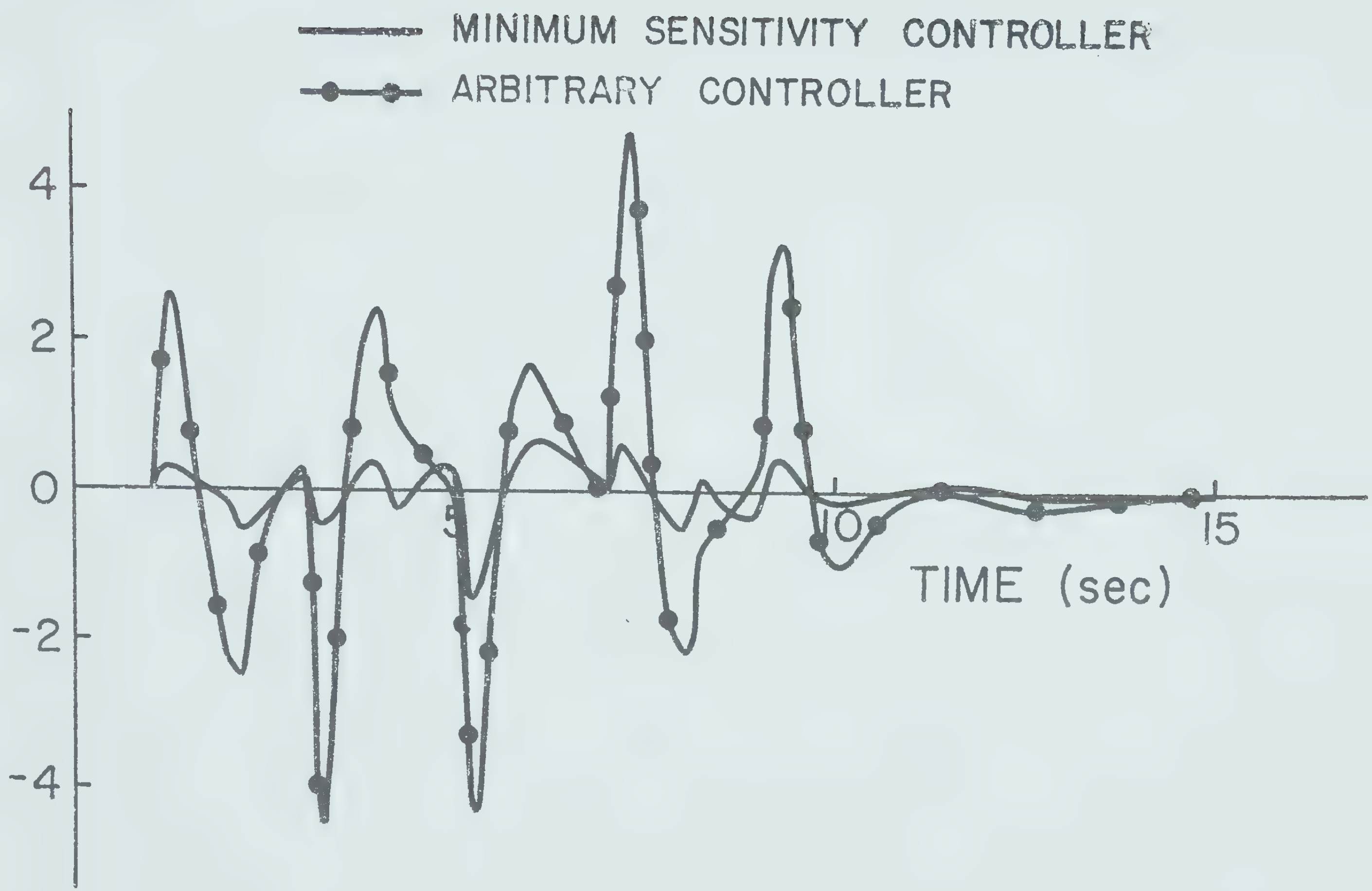

Figure 30. Disturbance Effect on X_2 when using Integral Controller and Flying in Steady State of One Hundred and Thirty-five Knots

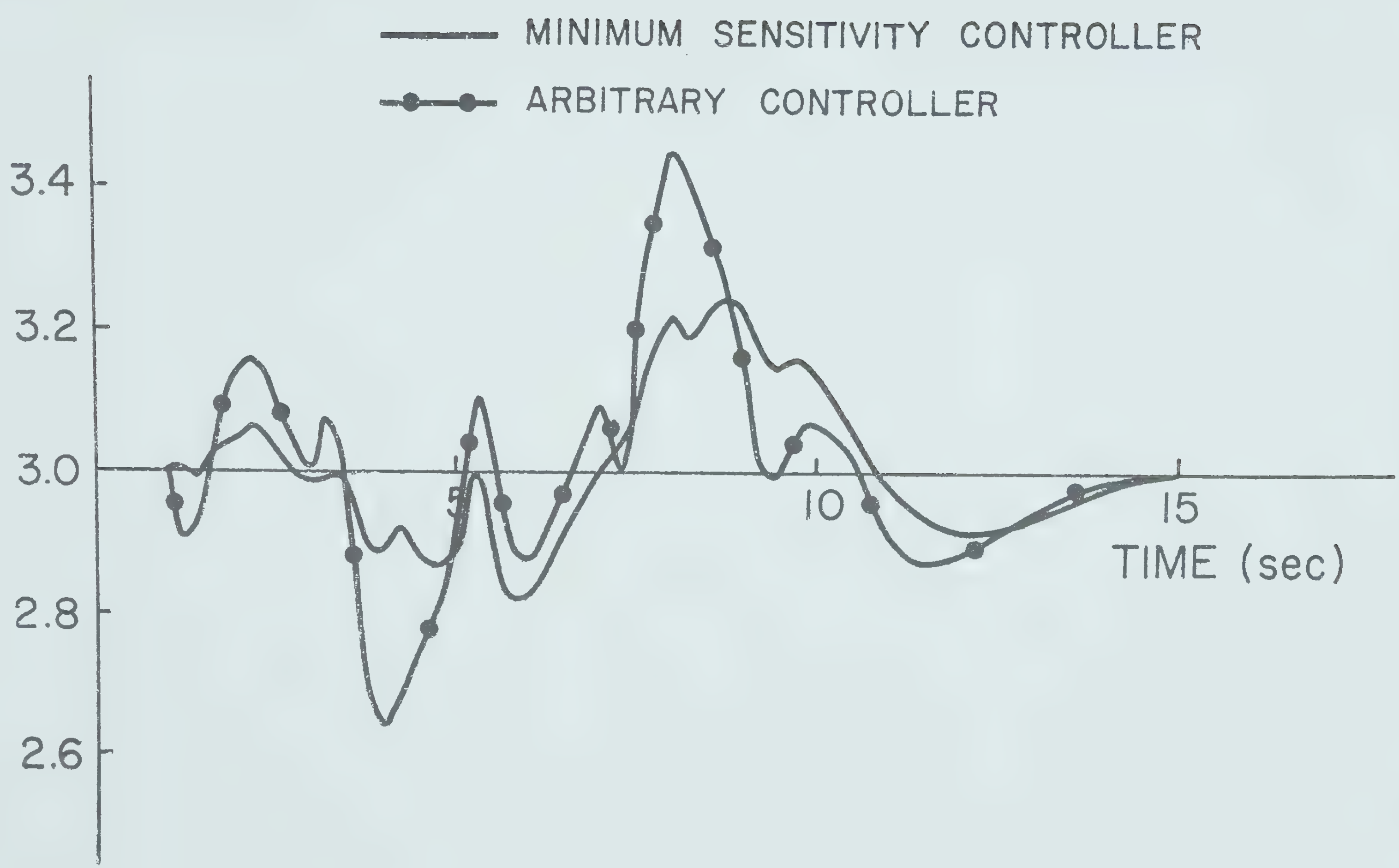

Figure 29. Disturbance Effect on X_1 when using Integral Controller and Flying in Steady State of One Hundred and Thirty-five Knots

CHAPTER (5)

SUMMARY AND CONCLUSIONS

5.1 Discussion of Results

A study of the results reveals that the minimum sensitivity design approach in constructing a constant gain feedback controller for a system has definite advantages over an arbitrary feedback controller when assigning the closed-loop poles of a system to specified locations in the complex frequency plane.

Since the location of the closed-loop poles of any system is an important factor determining the stability of the system, the sensitivity of these pole locations to variations in elements of the plant matrix are also critical. The sensitivity design approach ensures minimum variation of the pole locations when the element(s) of the plant matrix vary from their nominal values. This is illustrates quite effectively in chapters 2, 3 and 4.

The poles of a closed-loop system are also a factor in determining the response of the variables of a system. As can be seen by checking the responses in the previous three chapters, those obtained using the minimum sensitivity controller are much better than those using an arbitrary controller since the poles using the minimum sensitivity controller are more insensitive to change when the elements of the plant matrix vary, i.e. they tend to remain at the desired locations.

The results of chapter 4 further illustrate the effective use of the minimum sensitivity design approach. When using minimum sensitivity analysis, the effective incorporation of integral control to ensure zero

steady-state error of the selected variables and also decoupling of the modes in steady state, would suggest that the minimum sensitivity analysis procedure is not restricted in its applications but that it can quite possibly be used in conjunction with a number of existing control methods.

5.2 Possible Areas for Further Research

The research in this thesis is intended to show one possible method of designing a constant gain feedback controller for VTOL aircraft. The results indicate that the controller can be effective over the flight regime of a VTOL aircraft. The following are some areas for further research.

(1) Chapter 2, 3 and 4 took into account only the variations of the plant matrix A of the system. A further study could take into account the variations of the input matrix B and also variations in the feedback matrix K.

(2) In this thesis it was assumed all the state variables were available for feedback, and in many cases this may not be the case. A study could be conducted for the case where some of the states are not available for feedback. The incorporation of an observer (state reconstructor) to estimate the unmeasurable state variables could be considered. The minimum sensitivity design approach could then be used to design the feedback controller as before.

(3) The use of some other method other than unity-rank feedback could also be used to obtain the minimum sensitivity controller.

BIBLIOGRAPHY

1 Narendra, K.S. and Tripathi, S.S., "Identification and Optimization of Aircraft Dynamics", Journal of Aircraft, Vol. 10, No. 6, April 1973.

2 Murphy, R.D. and Narendra, K.S., "Design of Helicopter Stabilization Systems Using Optimal Control Theory", Journal of Aircraft, Vol. 6, No. 2, March-April 1969.

3 Swaim, R.L., "Minimum Control Power for VTOL Aircraft Stability Augmentation", Journal of Aircraft, May-June 1970.

4 Miller, D.P. and Clark, J.W., "Research on VTOL Aircraft Handling Qualities Criteria", Journal of Aircraft, Vol. 2, No. 2, May-June 1965.

5 Mehra, R.K., Stepner, D.E. and Tyler, J.S., "Maximum Likelihood Identification of Aircraft Stability and Control Derivatives", Journal of Aircraft, Vol. 11, No. 2, February 1974.

6 Kaufman, L.A., "A Concept for the Development of a Universal Automatic Flight Control System for VTOL Aircraft", American Helicopter Society Journal, Vol. 10-12, 1965-1967.

7 Kanai, K., Nikiforuk, P.N. and Gupta, M.M., "Optimal Control System Synthesis with respect to Desired Response Characteristics and Available States as Applied to VTOL Aircraft", Joint Automatic Control Conference, University of Stanford, Session 9 - Flight Control, 1972.

8 Hiscocks, R.D., "STOL Aircraft - A Perspective", The Aeronautical Journal of the Royal Aeronautical Society, Vol. 72, January 1968.

9 Carlock, G.W. and Sage, A.P., "VTOL Flight-Control System Design Using Sensitivity Analysis", IEEE Transactions on Aerospace and Electronic Systems, Vol. AES-11, No. 2, March 1975.

10 Buffum, P.S. and Robertson, W.T., "A Hover Augmentation System for Helicopters", Journal of Aircraft, Vol. 4, No. 4, July-August 1967.

11 Bischoff, D., Duffy, R. and Kaufman, H., "Automatic Control of Adverse Yaw in the Landing Environment using Optimal Control Theory", Journal of Aircraft, Vol. 11, No. 9, September 1974.

12 AIAA VTOL Systems Committee, "VTOL - 1968", Journal of Aircraft, Vol. 6, No. 4, July-August 1969.

13 Anderson, S.B., "Stability and Control Harmony in Approach and Landing", AGARD Conference Proceedings, No. 160, January 1975.

14 Crossley, T.R. and Porter, B., "Synthesis of Helicopter Stabilization System Using Modal Control Theory", American Institute of Aeronautics and Astronautics, AIAA Paper, No. 70-1036, August 1970.

15 Shapiro, J., "The Helicopter", The MacMillan Company, New York, 1st Edition, 1960.

16 Young, P.C. and Willems, Y.C., "An Approach to the Linear Multivariable Seromechanism Problem", International Journal on Control, Vol. 15, No. 5, 1972.

17 Gourishankar, V. and Ramar, K., "Control of Water Quality in Polluted Streams", International Journal of System Sciences, In Press, 1977.

18 Gourishankar, V. and Ramar, K., "Utilization of the Design Freedom of Pole Assignment Feedback Controllers of Unrestricted Rank", International Journal on Control, Vol. 24, No. 3, 1976.

19 Ramaswami, B. and Ramar, K., "Transformation to the Phase-Variable Canonical Form", IEEE Transactions on Automatic Control, Vol. AC-13, No. 6, December 1968.

20 Bhattacharyya, S.P. and Del Nero Gomes, A.C., "Output Controllability And Eigenvalue Assignability in Single-Output Linear Systems", IEEE Transactions on Automatic Control, Vol. AC-18, pp. 540-541, October 1973.

21 Davison, E.J. and Goldberg, R.W., "A Design Technique for the Incomplete State Feedback Problem in Multivariable Control Systems", Automatica, Vol. 5, pp. 335-346, 1969.

22 Fallside, F. and Patel, R.V., "Pole And Zero Assignment for Linear Multivariable Systems Using Unity-Rank Feedback", Electronics Letters, Vol. 8, No. 13, July 1971.

23 Hannah, R.S. and Mee, D.H., "Use of Unity-Rank Output Feedback to Guide the Selection of Feedback Coefficients", Electronics Letters, Vol. 4, No. 12, June 1973.

24 Gourishankar, V. and Ramar, K., "Pole Assignment with Minimum Eignevalue Sensitivity to Plant Parameter Variations", International Journal on Control, Vol. 23, No. 4, April 1976.

25 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966.

26 Morgan, Jr., B.S., "Sensitivity Analysis and Synthesis of Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 3, July 1966.

27 Rane, D.S., "A Simplified Transformation to Phase-Variable or Canonical Form", IEEE Transactions on Automatic Control, Vol. AC-11, No. 3, July 1966.

28 Rosenbrock, H.H. and Rowe, A., " Allocation of Poles and Zeros", Proceedings IEE, Vol. 117, No. 9, Sept 1970.

29 Seraji, H., "Design of Multivariable Systems Using Unity-Rank State Feedback: Further Results", Electronics Letters, Vol. 11, No. 2, January 1975.

30 Seraji, H., "Restriction on Attainable Poles and Methods for Pole Assignment with Output Feedback", Proceedings IEE, Vol. 121, No. 3, March 1974.

31 Seraji, H., "Unattainability of Certain Pole Positions in Single-input Systems with Output Feedback", International Journal on Control, Vol. 22, No. 1, March 1975.

32 Seraji, H., "Effect of Unity-Rank Feedback on the Transfer-function Matrix of A Multivariable System", Electronics Letters, Vol. 9, No. 20, October 1973.

33 Shah, S.L., Fisher, D.G. and Seborg, D.E., "Eigenvalue/Eigenvector Assignment for Multivariable Systems and Further Results for Output Feedback Control", Electronics Letters, Vol. 11, No. 16, August 1975.

34 Solheim, O.A., "Design of Optimal Control Systems with Prescribed Eigenvalues", International Journal on Control, Vol. 15, No. 1, 1972.

35 Singer, R.A., "Selecting State Variables to Minimize Eigenvalue Sensitivity of Multivariable Systems", Automatica, Vol. 5, pp. 85-93, 1968

36 Wilkie, D.F. and Perkins, W.R., "Design of Model Following Systems Using the Companion Transformation", Automatica, Vol. 5, pp. 615-622, 1969.

37 Tiroshi, I. and Elliott, J.R., "Explicit Model Following Control Scheme Incorporating Integral Control", Journal of Aircraft, Vol. 11, No. 6, June 1974.

38 Perkins, C.D., "Development of Airplane Stability And Control Technology", Journal of Aircraft, Vol. 7, No. 4, July-August 1970.

39 Gablehouse, C., "Helicopters and Autogiros", J.B. Lippincott Company Philadelphia and New York, Revised Edition, 1969.

40 Young, R.A., "Helicopter Engineering", The Ronald Press Company New York, Copywrite, 1949.

41 Scolatti, Colonel, C.A., "Progress of the USAF Inflight Program: Low Speed Control to Landing on Instruments in Helicopters", AGARD Conference Proceedings, No. 86, pp. 5-1 - 5-7, June 1971.

APPENDIX I

Fortran Program Used to do Design Calculations

```
C   THE FOLLOWING PROGRAM DOES ALL THE DESIGN CALCULATIONS
C      TO DETERMINE THE REQUIRED MINIMUM SENSITIVITY FEEDBACK
C      CONTROLLER FOR THE 4TH-ORDER VTOL AIRCRAFT SYSTEM UNDER
C      STUDY.  THE PROGRAM UTILIZES THE IMSL SUBROUTINE ZXMIN
C      TO FIND THE MINIMUM OF THE SENSITIVITY FUNCTIONAL J.
      EXTERNAL FUNCT
      DOUBLE PRECISION A1(4,4),A2(4,4),B(4,2),R(2,1),AA(4,4)
     +,AAA(4,4),QQ(4,4),P(4,4),PINV(4,4),ALF(4),BQ(4,1)
     +,A01(4,4),PP(4,4),A001(4,4),ABK(4,4),AXX(4,4),BKK(2,4)
     +,BK(4,4),A1K(4,4),R0(4,4),R1(4,4),R2(4,4),R3(4,4)
     +,Q(4,4),QINV(4,4),Q1(4,1),Q2(4,1),Q3(4,1),P1(1,4)
     +,P2(1,4),P3(1,4),P4(1,4),A0(4,4),A00(4,4),WKAREA(28)
      DOUBLE PRECISION X(1),H(1),G(1),W(3)
      COMPLEX*16 X1,X2,X3,X4,X5,X6,X7,X8,X9,X10,X11,X12,T2
     +,SA1(4,4),SA2(4,4),SA3(4,4),SA4(4,4),D01(4,4)
     +,D02(4,4),D03(4,4),D04(4,4),DS1(4,4),DS2(4,4),DS3(4,4)
     +,DS4(4,4),S(4),GT(4),SUMM
      COMMON A1,A2,B,ALF,DS1,DS2,DS3,DS4,S,X
      INTEGER V,Y,U,E,FF
      READ(5,10)((A1(I,J),J=1,4),I=1,4)
      READ(5,10)((A2(I,J),J=1,4),I=1,4)
      READ(5,15)((B(I,J),J=1,2),I=1,4)
      READ(5,51)(ALF(I),I=1,4)
      READ(5,52)(S(I),I=1,4)
      READ(5,1500)X(1)
      READ(5,10)((A01(I,J),J=1,4),I=1,4)
   10 FORMAT(4D15.5)
 1500 FORMAT(D10.1)
   15 FORMAT(2D15.5)
   51 FORMAT(4D15.3)
   52 FORMAT(2D10.3)
      CALL VMULFF(A1,A2,4,4,4,4,4,AA,4,IER)
      CALL VMULFF(A1,AA,4,4,4,4,4,AAA,4,IER)
      DO1I=1,4
      DO1J=1,4
      IF(I.EQ.J)GOTO3
      R0(I,J)=0.D0
      GO TO 1
    3 R0(I,J)=1.D0
    1 CONTINUE
      X1=S(1)
      X2=S(1)**2
```

```
      X3=S(1)**3
      X4=S(2)
      X5=S(2)**2
      X6=S(2)**3
      X7=S(3)
      X8=S(3)**2
      X9=S(3)**3
      X10=S(4)
      X11=S(4)**2
      X12=S(4)**3
      DO4I=1,4
      CALL GADJT(R0,A01,R1,R0,4,2)
      CALL GADJT(R1,A01,R2,R0,4,3)
      CALL GADJT(R2,A01,R3,R0,4,4)
      IF(I.EQ.1)GOTO67
      IF(I.EQ.2)GOTO68
      IF(I.EQ.3)GOTO71
      IF(I.EQ.4)GOTO72
   67 GG=0.
      DO69J=1,4
      DO69K=1,4
   69 DS1(J,K)=X3*R0(J,K)+X2*R1(J,K)+X1*R2(J,K)+R3(J,K)
      GO TO 4
   68 GG=0.
      DO73J=1,4
      DO73K=1,4
   73 DS2(J,K)=X6*R0(J,K)+X5*R1(J,K)+X4*R2(J,K)+R3(J,K)
      GO TO 4
   71 GG=0.
      DO66J=1,4
      DO66K=1,4
   66 DS3(J,K)=X9*R0(J,K)+X8*R1(J,K)+X7*R2(J,K)+R3(J,K)
      GO TO 4
   72 GG=0.
      DO74J=1,4
      DO74K=1,4
   74 DS4(J,K)=X12*R0(J,K)+X11*R1(J,K)+X10*R2(J,K)+R3(J,K)
    4 CONTINUE
      DO53I=1,4
   53 GT(I)=4.*S(I)**3+3.*ALF(4)*S(I)**2+2.*ALF(3)*S(I)+ALF(2)
      N=1
      NSIG=3
      MAXFN=500
```

```
      IOPT=0
      CALL ZXMIN(FUNCT,N,NSIG,MAXFN,IOPT,X,H,G,F,W,IER)
      WRITE(6,990)MAXFN
  990 FORMAT(40X,8HMAXFN = ,I4)
      WRITE(6,95)X(1),F,G(1)
   95 FORMAT(1H-,30HTHE MINIMUM VALUE OF Q(2,1) = ,
     +D13.5/20X,29HTHE SENSITIVITY FUNCTION J = ,D13.5
     +/20X,24HTHE GRADIENT VECTOR G = ,D13.5)
      R(1,1)=1.D0
      R(2,1)=X(1)
  385 WRITE(6,43)
   43 FORMAT(1H-,63X,1HT)
      WRITE(6,44)R(1,1),R(2,1)
   44 FORMAT(34X,10HQ(I,J) = {,D12.3,D12.3,4H   })
      CALL VMULFF(B,R,4,2,1,4,2,BQ,4,IER)
      CALL VMULFF(A1,BQ,4,4,1,4,4,Q1,4,IER)
      CALL VMULFF(AA,BQ,4,4,1,4,4,Q2,4,IER)
      CALL VMULFF(AAA,BQ,4,4,1,4,4,Q3,4,IER)
      DO25J=1,4
      Q(J,1)=BQ(J,1)
      Q(J,2)=Q1(J,1)
      Q(J,3)=Q2(J,1)
      Q(J,4)=Q3(J,1)
   25 CONTINUE
      DO65J=1,4
      DO65I=1,4
   65 QQ(I,J)=Q(I,J)
      CALL LINV2F(QQ,4,4,QINV,4,WKAREA,IER)
      Y=1
      U=4
      DO70J=1,4
   70 P1(Y,J)=QINV(U,J)
      CALL VMULFF(P1,A1,1,4,4,1,4,P2,1,IER)
      CALL VMULFF(P1,AA,1,4,4,1,4,P3,1,IER)
      CALL VMULFF(P1,AAA,1,4,4,1,4,P4,1,IER)
      DO75I=1,4
      P(1,I)=P1(1,I)
      P(2,I)=P2(1,I)
      P(3,I)=P3(1,I)
      P(4,I)=P4(1,I)
   75 CONTINUE
      DO115J=1,4
      DO115I=1,4
```

```
115 PP(I,J)=P(I,J)
    CALL LINV2F(PP,4,4,PINV,4,WKAREA,IER)
    CALL VMULFF(A1,PINV,4,4,4,4,4,A00,4,IER)
    CALL VMULFF(P,A00,4,4,4,4,4,A0,4,IER)
    DO7V=1,4
    IF(V.EQ.1)GOTO8
    IF(V.EQ.2)GOTO9
    IF(V.EQ.3)GOTO11
    IF(V.EQ.4)GOTO56
  8 ZX=0.
    DO12J=1,4
    DO13E=1,4
    SUMM=(0.D0,0.D0)
    DO14K=1,4
 14 SUMM=SUMM+DS1(E,K)*P(K,J)
 13 DO1(E,J)=SUMM
 12 CONTINUE
    DO16J=1,4
    DO17E=1,4
    SUMM=(0.D0,0.D0)
    DO538K=1,4
538 SUMM=SUMM+PINV(E,K)*DO1(K,J)
 17 SA1(J,E)=SUMM/GT(V)
 16 CONTINUE
    GO TO 7
  9 ZX=0.
    DO18J=1,4
    DO19E=1,4
    SUMM=(0.D0,0.D0)
    DO21K=1,4
 21 SUMM=SUMM+DS2(E,K)*P(K,J)
 19 DO2(E,J)=SUMM
    DO22FF=1,4
    SUMM=(0.D0,0.D0)
    DO23K=1,4
 23 SUMM=SUMM+PINV(FF,K)*DO2(K,J)
 22 SA2(J,FF)=SUMM/GT(V)
 18 CONTINUE
    GO TO 7
 11 ZX=0.
    DO24J=1,4
    DO26E=1,4
    SUMM=(0.D0,0.D0)
```

```
      DO27K=1,4
   27 SUMM=SUMM+DS3(E,K)*P(K,J)
   26 D03(E,J)=SUMM
      DO28FF=1,4
      SUMM=(0.D0,0.D0)
      DO29K=1,4
   29 SUMM=SUMM+PINV(FF,K)*D03(K,J)
   28 SA3(J,FF)=SUMM/GT(V)
   24 CONTINUE
      GO TO
   56 ZX=0.
      DO31J=1,4
      DO32E=1,4
      SUMM=(0.D0,0.D0)
      DO33K=1,4
   33 SUMM=SUMM+DS4(E,K)*P(K,J)
   32 D04(E,J)=SUMM
      DO34FF=1,4
      SUMM=(0.D0,0.D0)
      DO54K=1,4
   54 SUMM=SUMM+PINV(FF,K)*D04(K,J)
   34 SA4(J,FF)=SUMM/GT(V)
   31 CONTINUE
    7 CONTINUE
      T2=SA1(1,1)**2+SA2(1,1)**2+SA3(1,1)**2+SA4(1,1)**2
     ++SA1(1,2)**2+SA2(1,2)**2+SA3(1,2)**2+SA4(1,2)**2
     ++SA1(2,1)**2+SA2(2,1)**2+SA3(2,1)**2+SA4(2,1)**2
     ++SA1(2,2)**2+SA2(2,2)**2+SA3(2,2)**2+SA4(2,2)**2
      TT=CDABS(T2)
      CALL VMULFF(A01,F,4,4,4,4,4,A001,4,IER)
      CALL VMULFF(PINV,A001,4,4,4,4,4,ABK,4,IER)
      DO7000I=1,4
      DO7000J=1,4
 7000 AXX(I,J)=ABK(I,J)-A1(I,J)
      DO300J=1,4
      BKK(1,J)=AXX(1,J)/BQ(1,1)
  300 BKK(2,J)=BKK(1,J)*R(2,1)
      WRITE(6,854)
      WRITE(6,855)((BKK(I,J),J=1,4),I=1,2)
  854 FORMAT('-',20X,'THE FEEDBACK MATRIX K IS')
  855 FORMAT(4D12.5)
      CALL VMULFF(B,BKK,4,2,4,4,2,BK,4,IER)
      DO7070I=1,4
```

```
      DO7070J=1,4
 7070 A1K(I,J)=A1(I,J)+BK(I,J)
      WRITE(6,860)
      WRITE(6,861)((A1K(I,J),J=1,4),I=1,4)
      WRITE(6,99)
      WRITE(6,901)TT
      STOP
  860 FORMAT('-',35X,'THE MATRIX A + BK IS')
  861 FORMAT(10X,4D12.5)
   99 FORMAT(1H-)
  901 FORMAT(1X,25HTHE MINIMUM VALUE OF J = ,D15.5)
      END
C   SUBROUTINE TO DETERMINE ADJ(SI - AO)
      SUBROUTINE GADJT(XI,XO,RR,RRR,L,N)
      REAL*8 XO(4,4),D(4,4),DD(4,4),RR(4,4),XI(4,4),C(4,4)
     +,RRR(4,4)
      DO1J=1,L
      DO2I=1,L
      SUMM=0.DO
      DO3K=1,L
    3 SUMM=SUMM+XI(I,K)*XO(K,J)
    2 C(I,J)=SUMM
    1 CONTINUE
      ALA=0.DO
      DO15I=1,L
   15 ALA=ALA-C(I,I)
      ALF=ALA/(N-1)
      DO4J=1,L
      DO5I=1,L
      SUM=0.DO
      DO6K=1,L
    6 SUM=SUM+XO(I,K)*XI(K,J)
    5 D(I,J)=SUM
    4 CONTINUE
      DO7J=1,L
      DO7I=1,L
    7 DD(I,J)=ALF*RRR(I,J)
      DO8I=1,L
      DO8J=1,L
    8 RR(I,J)=D(I,J)+DD(I,J)
      RETURN
      END
```

```
C     SUBROUTINE TO FIND THE MINIMUM FUNCTION J
C
      SUBROUTINE FUNCT(N,X,F)
      DOUBLE PRECISION RZ(2,1),AAZ(4,4),AAAZ(4,4),QQZ(4,4)
     +,PZ(4,4),PINVZ(4,4),BQZ(4,1),PPZ(4,4),QZ(4,4),QINVZ(4,4)
     +,Q1Z(4,1),Q2Z(4,1),Q3Z(4,1),P1Z(1,4),P2Z(1,4),P3Z(1,4)
     +,P4Z(1,4),AOZ(4,4),AOOZ(4,4),WKAREA(28),X(N),A1(4,4)
     +,A2(4,4),B(4,2),ALF(4)
      COMPLEX*16 T1,SA1Z(4,4),SA2Z(4,4),SA3Z(4,4),SA4Z(4,4)
     +,DO1Z(4,4),DO2Z(4,4),DO3Z(4,4),DO4Z(4,4),S(4),GTZ(4)
     +,SUMM,DS1(4,4),DS2(4,4),DS3(4,4),DS4(4,4)
      COMMON A1,A2,B,ALF,DS1,DS2,DS3,DS4,S
      INTEGER Y,U,E,FF,V
      RZ(1,1)=1.D0
      RZ(2,1)=X(1)
      WRITE(6,999)X(N)
  999 FORMAT(F20.5)
      CALL VMULFF(A1,A2,4,4,4,4,4,AAZ,4,IER)
      CALL VMULFF(A1,AAZ,4,4,4,4,4,AAAZ,4,IER)
      CALL VMULFF(B,RZ,4,2,1,4,2,BQZ,4,IER)
      CALL VMULFF(A1,BQZ,4,4,1,4,4,Q1Z,4,IER)
      CALL VMULFF(AAZ,BQZ,4,4,1,4,4,Q2Z,4,IER)
      CALL VMULFF(AAAZ,BQZ,4,4,1,4,4,Q3Z,4,IER)
      DO997V=1,4
  997 GTZ(V)=4.D0*S(V)**3+3.D0*ALF(4)*S(V)**2+2.D0*ALF(3)*
     +S(V)+ALF(2)
      DO25J=1,4
      QZ(J,1)=BQZ(J,1)
      QZ(J,2)=Q1Z(J,1)
      QZ(J,3)=Q2Z(J,1)
      QZ(J,4)=Q3Z(J,1)
   25 CONTINUE
      DO65J=1,4
      DO65V=1,4
   65 QQZ(V,J)=QZ(V,J)
      CALL LINV2F(QQZ,4,4,QINVZ,4,WKAREA,IER)
      Y=1
      U=4
      DO70J=1,4
   70 P1Z(Y,J)=QINVZ(U,J)
      CALL VMULFF(P1Z,A1,1,4,4,1,4,P2Z,1,IER)
      CALL VMULFF(P1Z,AAZ,1,4,4,1,4,P3Z,1,IER)
      CALL VMULFF(P1Z,AAAZ,1,4,4,1,4,P4Z,1,IER)
```

```
      DO75V=1,4
      PZ(1,V)=P1Z(1,V)
      PZ(2,V)=P2Z(1,V)
      PZ(3,V)=P3Z(1,V)
      PZ(4,V)=P4Z(1,V)
   75 CONTINUE
      DO115J=1,4
      DO115V=1,4
  115 PPZ(V,J)=PZ(V,J)
      CALL LINV2F(PPZ,4,4,PINVZ,4,WKAREA,IER)
      CALL VMULFF(A1,PINVZ,4,4,4,4,4,A00Z,4,IER)
      CALL VMULFF(PZ,A00Z,4,4,4,4,4,A0Z,4,IER)
      DO7V=1,4
      IF(V.EQ.1)GOTO8
      IF(V.EQ.2)GOTO9
      IF(V.EQ.3)GOTO11
      IF(V.EQ.4)GOTO56
    8 ZX=0.
      DO12J=1,4
      DO13E=1,4
      SUMM=0.D0
      DO14K=1,4
   14 SUMM=SUMM+DS1(E,K)*PZ(K,J)
   13 D01Z(E,J)=SUMM
   12 CONTINUE
      DO16J=1,4
      DO17E=1,4
      SUMM=0.D0
      DO538K=1,4
  538 SUMM=SUMM+PINVZ(E,K)*D01Z(K,J)
   17 SA1Z(J,E)=SUMM/GTZ(1)
   16 CONTINUE
      GO TO 7
    9 ZX=0.
      DO18J=1,4
      DO19E=1,4
      SUMM=0.D0
      DO21K=1,4
   21 SUMM=SUMM+DS2(E,K)*PZ(K,J)
   19 D02Z(E,J)=SUMM
      DO22FF=1,4
      SUMM=0.D0
      DO23K=1,4
```

```
 23 SUMM=SUMM+PINVZ(FF,K)*DO2Z(K,J)
 22 SA2Z(J,FF)=SUMM/GTZ(2)
 18 CONTINUE
    GO TO 7
 11 ZX=0.
    DO24J=1,4
    DO26E=1,4
    SUMM=0.D0
    DO27K=1,4
 27 SUMM=SUMM+DS3(E,K)*PZ(K,J)
 26 DO3Z(E,J)=SUMM
    DO28FF=1,4
    SUMM=0.D0
    DO29K=1,4
 29 SUMM=SUMM+PINVZ(FF,K)*DO3Z(K,J)
 28 SA3Z(J,FF)=SUMM/GTZ(3)
 24 CONTINUE
    GO TO 7
 56 ZX=0.
998 FORMAT(2D15.5)
    DO31J=1,4
    DO32E=1,4
    SUMM=0.D0
    DO33K=1,4
 33 SUMM=SUMM+DS4(E,K)*PZ(K,J)
 32 DO4Z(E,J)=SUMM
    DO34FF=1,4
    SUMM=0.D0
    DO54K=1,4
 54 SUMM=SUMM+PINVZ(FF,K)*DO4Z(K,J)
 34 SA4Z(J,FF)=SUMM/GTZ(4)
 31 CONTINUE
  7 CONTINUE
    T1=SA1Z(1,1)**2+SA2Z(1,1)**2+SA3Z(1,1)**2+SA4Z(1,1)**2
   ++SA1Z(1,2)**2+SA2Z(1,2)**2+SA3Z(1,2)**2+SA4Z(1,2)**2
   ++SA1Z(2,1)**2+SA2Z(2,1)**2+SA3Z(2,1)**2+SA4Z(2,1)**2
   ++SA1Z(2,2)**2+SA27(2,2)**2+SA3Z(2,2)**2+SA4Z(2,2)**2
    F=CDABS(T1)
800 CONTINUE
    RETURN
    END
```

APPENDIX II

Continuous System Modeling Program (CSMP) Used For Simulations

```
LABEL HELICOPTER SYSTEM
INITIAL
UB1=NOMINAL PILOT COLLECTIVE INPUT.
UB2=NOMINAL PILOT LONGITUDINAL CYCLIC INPUT.
X10=INITIAL CONDITION FOR X1.
X20=INITIAL CONDITION FOR X2.
X30=INITIAL CONDITION FOR X3.
X40=INITIAL CONDITION FOR X4.
A=VALUE OF AT32 AT START OF SIMULATION.
B=VALUE OF AT34 AT START OF SIMULATION.
C=VALUE OF AT32 AT END OF SIMULATION.
D=VALUE OF AT34 AT END OF SIMULATION.
E=TIME WHEN AT32 AND AT34 ARE TO STOP VARYING.
DYNAMIC
T=RAMP(0.0)
Y1=0.02414*T+A
Y2=0.104016*T+B
AA32=FCNSW(T,0.,A,Y1)
AA34=FCNSW(T,0.,B,Y2)
T2=T-E
AT32=FCNSW(T2,AA32,C,C)
AT34=FCNSW(T2,AA34,D,D)
X1D=AT11*X1+AT12*X2+AT13*X3+AT14*X4+B11*UB1+B12*UB2
X2D=AT21*X1+AT22*X2+AT23*X3+AT24*X4+B21*UB1+B22*UB2
X3D=AT31*X1+AT32*X3+AT33*X3+AT34*X4+B31*UB1+B32*UB2
X4D=AT43*X3
X1=INTGRL(X10,X1D)
X2=INTGRL(X20,X2D)
X3=INTGRL(X30,X3D)
X4=INTGRL(X40,X4D)
TIMER DELT=0.01,FINTIM=20.0,PRDEL=0.2
PRINT X1,X2,X3,X4
END
STOP
ENDJOB
```

APPENDIX III

<u>Calculation of Percent(%) Change in Pole Locations</u>

Suppose the pole locations before variations in the system matrix are as shown in the complex frequency plane of figure 1.

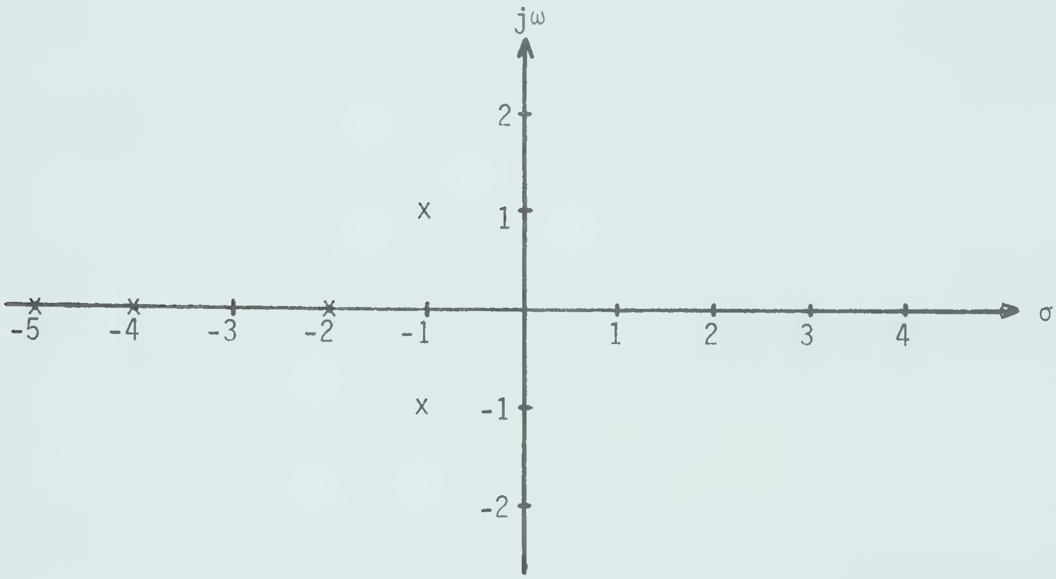

Figure A-1. Pole Locations

Before variation we will denote the poles as

$$s_1 = -5$$

$$s_2 = -4$$

$$s_3 = -2$$

$$s_4 = -1 + j1$$

$$s_5 = =1 - j1$$

Due to some change in the system matrix the pole locations change and the new pole positions are represented as

$$s_1^* = -5.5$$

$$s_2^* = -3.4 + j2$$

$$s_3^* = -3.4 - j2$$

$$s_4^* = 1.0 + j0.5$$

$$s_5^* = 1.0 - j0.5$$

which indicate a change in the pole locations of

$$\Delta s_1 = -0.5$$

$$\Delta s_2 = 0.6 + j2$$

$$\Delta s_3 = -1.4 - j2$$

$$\Delta s_4 = 2.0 - j0.5$$

$$\Delta s_5 = 2.0 + j0.5$$

The above changes are calculated by the equation

$$\Delta s_i = s_i^* - s_i \quad ; \quad i = 1,2, \ldots ,n$$, where n = no. of poles

The percent change in s_i is then calculated by

$$\% \text{ change of } s_i = \frac{|\Delta s_i|}{|s_i|} \times 100 \quad ; i = 1,n \quad \text{-------} \quad 1$$

Now if there is a change in the angle of the pole location as well, it was calculated as

$$\Delta \underline{/s_i} = \underline{/s_i^*} - \underline{/s_i} \quad ; \quad i = 1,n \quad \text{------------} \quad 2$$

A positive angle was taken as shown below in figure 2.

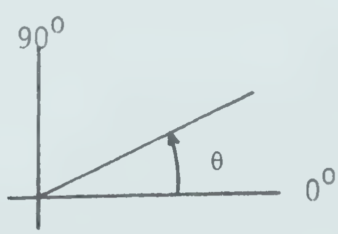

Figure A-2. Positive Angle

Thus using equations 1 and 2 we have for Δs_i , $i = 1,5$

$$\Delta s_1 = \frac{|\Delta s_1|}{|s_1|} \times 100 = \frac{0.5}{5} \times 100 = 10\% \quad @ \; \underline{/\ 0^0}$$

$$\Delta s_2 = \frac{|\Delta s_2|}{|s_2|} \times 100 = \frac{2.09}{4} \times 100 = 52\% \quad @ \; \underline{/-30.5}^0$$

$$\Delta s_3 = \frac{|\Delta s_3|}{|s_3|} \times 100 = \frac{2.44}{2} \times 100 = 122\% \quad @ \; \underline{/\ 30.5}^0$$

$$\Delta s_4 = \frac{|\Delta s_4|}{|s_4|} \times 100 = \frac{2.06}{1.41} \times 100 = 145.8\% \; @ \; \underline{/-108.4}^0$$

$$\Delta s_5 = \frac{|\Delta s_5|}{|s_5|} \times 100 = \frac{2.06}{1.41} \times 100 = 145.8\% \; @ \; \underline{/108.4}^0$$